Mathematics. — Prof. KLUYVER presents a paper: *"Evaluation of two definite integrals."*

Supposing x to be real the integrals

$$f(x, m) = \int_0^\infty \frac{\cos xt}{(1+t^2)^m}\, dt \quad \text{and} \quad \varphi(x, m) = \int_0^\infty \frac{\sin xt}{(1+t^2)^m}\, dt$$

will have a definite meaning, if only the real part of the parameter m be positive. In what follows we will show how to expand these integrals into rapidly converging power series.

The first of them, the integral $f(x, m)$, is a particular solution of the linear equation

$$x\frac{d^2y}{dx^2} - 2(m-1)\frac{dy}{dx} - xy = 0,$$

the primitive of which, involving two arbitrary constants A and B, may be written in the form

$$y = A\, L(x, m) + Bx^{2m-1} M(x, m),$$

where

$$L(x, m) = \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h}}{h!\, \Gamma(-m + \frac{3}{2} + h)},$$

$$M(x, m) = \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h}}{h!\, \Gamma(m + \frac{1}{2} + h)},$$

and the constants A and B must now be determined so that y represents the function $f(x, m)$. To find A, we suppose $m > \frac{1}{2}$ and put x equal to zero. In that case we have

$$f(0, m) = \int_0^\infty \frac{dt}{(1+t^2)^m} = \frac{\Gamma(\frac{1}{2})\, \Gamma(m-\frac{1}{2})}{2\, \Gamma(m)} = A\, L(0, m) = \frac{A}{\Gamma(-m + \frac{3}{2})}$$

and hence

$$A = -\frac{\pi}{2 \cos \pi m} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(m)}.$$

For the deduction of the constant B it is convenient to consider first the function $f(x, m)$ in another form. Let the real part of m still be positive, then we have

$$\frac{\Gamma(m)}{(1+t^2)^m}=\int_0^\infty e^{-u(1+t^2)}\,u^{m-1}\,du,$$

and hence

$$\Gamma(m)\,f(x,m)=\int_0^\infty e^{-u}\,u^{m-1}\,du\int_0^\infty e^{-ut^2}\cos xt\,dt=\tfrac{1}{2}\sqrt{\pi}\int_0^\infty e^{-u-\frac{x^2}{4u}}\,u^{m-\frac{3}{2}}\,du.$$

From the latter integral a simple functional relation is derivable. Changing the variable u into $\frac{x^2}{4v}$ we may write

$$\Gamma(m)\,f(x,m)=\frac{1}{2}\sqrt{\pi}\left(\frac{x}{2}\right)^{2m-1}\int_0^\infty e^{-v-\frac{x^2}{4v}}\,v^{-m-\frac{1}{2}}\,dv=$$

$$=\left(\frac{x}{2}\right)^{2m-1}\Gamma(1-m)\,f(x,1-m)$$

and so it follows that the function

$$\frac{\Gamma(m)f(x,m)}{\left(\frac{x}{2}\right)^m}=-\frac{\pi}{2\cos\pi m}\,\Gamma\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{-m}L(x,m)+$$

$$+\,2^{2m-1}B\,\Gamma(m)\left(\frac{x}{2}\right)^{m-1}M(x,m)$$

remains unaltered, if m is replaced by $1-m$.

Now obviously the series L and M are connected by the relation

$$L(x,1-m)=M(x,m),$$

hence we must have

$$2^{2m-1}B\,\Gamma(m)=+\frac{\pi}{2\cos\pi m}\,\Gamma\left(\frac{1}{2}\right),$$

$$B=\frac{\pi}{2\cos\pi m}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(m)}\left(\frac{1}{2}\right)^{2m-1},$$

and therefore

$$f(x,m)=\int_0^\infty\frac{\cos xt}{(1+t^2)^m}\,dt=\frac{\pi}{2\cos\pi m}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(m)}\left\{-L(x,m)+\left(\frac{x}{2}\right)^{2m-1}M(x,m)\right\}.$$

Now it will be observed, that the series $L(x,m)$ and $M(x,m)$ converge for all values of x and m, and so we must conclude, that the function $f(x,m)$ exists over the whole x-plane, that its only singularities are $x=0$ and $x=\infty$, and that therefore the integral, we started with, represents the function in a very incomplete manner.

Numerical evaluation of the integral for not too large values of x offers no difficulties, as the series $L(x,m)$ and $M(x,m)$ converge rapidly. Because of the equation

$$\Gamma(m) f(x, m) = \frac{1}{2} \sqrt{\pi} \int_0^\infty e^{-u-\frac{x^2}{4u}} u^{m-\frac{3}{2}} du$$

the result will always be a positive number and the integral will not vanish for any real value of x.

A few further remarks may be made. Firstly we may state, that $f(x, m)$ is intimately connected with BESSEL's function $J_n(x)$. In fact, by means of the usual expansion of $J_n(x)$ we may verify the relation

$$f(x, m) = -\frac{\pi}{2\cos \pi m} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(m)} \left(\frac{x}{2}\right)^{m-\frac{1}{2}} \left\{ e^{-\frac{\pi i}{2}\left(m-\frac{1}{2}\right)} J_{-m+\frac{1}{2}}\left(x e^{-\frac{\pi i}{2}}\right) - \right.$$

$$\left. - e^{+\frac{\pi i}{2}\left(m-\frac{1}{2}\right)} J_{m-\frac{1}{2}}\left(x e^{-\frac{\pi i}{2}}\right) \right\}.$$

From this we infer, that for positive integer values of m the origin $x = 0$ ceases to be a singular point, and that $f(x, m)$ can be expressed in finite terms. We shall find by actual substitution of the finite expressions for $J_{-m+\frac{1}{2}}\left(x e^{-\frac{\pi i}{2}}\right)$ and $J_{m-\frac{1}{2}}\left(x e^{-\frac{\pi i}{2}}\right)$

$$f(x, m) = \frac{\pi}{2} \cdot \frac{e^{-x}}{(m-1)!} \left(\frac{x}{2}\right)^{m-1} \sum_{h=0}^{h=m-1} \frac{(m-1+h)!}{h!(m-1-h)!} \cdot \left(\frac{1}{2x}\right)^h.$$

However this result may be obtained in a simpler way as follows. It can be shewn, that $f(x, m)$ obeys the relation

$$D^h_{\alpha=1} \left\{ \alpha^{-m+\frac{1}{2}} f(x\sqrt{\alpha}, m) \right\} = \frac{(-1)^h \Gamma(m+h)}{\Gamma(m)} f(x, m+h),$$

and since we have

$$f(x\sqrt{\alpha}, 1) = \int \frac{\cos xt \sqrt{\alpha}}{1+t^2} dt = \frac{\pi}{2} e^{-x\sqrt{\alpha}},$$

we get for all positive values of m

$$f(x, m) = \frac{\pi}{2} \cdot \frac{(-1)^{m-1}}{(m-1)!} D^{m-1}_{\alpha=1} \left\{ \frac{e^{-x\sqrt{\alpha}}}{\sqrt{\alpha}} \right\},$$

a result that can be identified with that obtained before.

The singularity in the origin $x = 0$ becomes logarithmic, when $2m$ is an odd integer $2k + 1$. The expression of $f(a, m)$ is in this

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case somewhat intricate. By repeated differentiations it is derived from $f\left(x, \frac{1}{2}\right)$, for we have

$$D^k_{x=1}\, f\left(x \sqrt{a}, \frac{1}{2}\right) = \frac{(-1)^k \Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} f\left(x, k + \frac{1}{2}\right).$$

To evaluate $f\left(x, \frac{1}{2}\right)$, we put $m = \frac{1}{2} + \sigma$ in the general expression for $f(x, m)$, and make σ tend to zero. In this way we get

$$f\left(x, \frac{1}{2}\right) = \lim_{\sigma=0} \frac{\pi}{2 \sin \pi\sigma} \left\{ \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h}}{h!\,\Gamma(h+1-\sigma)} - \left(\frac{x}{2}\right)^{2\sigma} \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h}}{h!\,\Gamma(h+1+\sigma)} \right\},$$

$$f\left(x, \frac{1}{2}\right) = \int_0^\infty \frac{\cos xt}{\sqrt{1+t^2}}\, dt = \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h}}{(h!)^2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{h} - C - \log \frac{x}{2}\right).$$

We shall now pass on to consider the second integral

$$\varphi(x, m) = \int_0^\infty \frac{\sin xt}{(1+t^2)^m}\, dt,$$

and it will appear that its character is quite similar to that of the first.

Again we transform the integral by the aid of the identity

$$\frac{\Gamma(m)}{(1+t^2)^m} = \int_0^\infty e^{-u(1+t^2)}\, u^{m-1}\, du$$

and obtain

$$\Gamma(m)\, \varphi(x, m) = \int_0^\infty e^{-u}\, u^{m-1}\, du \int_0^\infty e^{-ut^2} \sin xt\, dt.$$

A further transformation gives

$$\int_0^\infty e^{-ut^2} \sin xt\, dt = \frac{x}{2u} \int_0^1 e^{-\frac{x^2}{4u}(1-t^2)}\, dt = \frac{x}{4u} \int_0^1 e^{-\frac{x^2 w}{4u}} (1-w)^{-\frac{1}{2}}\, dw.$$

and therefore

$$\Gamma(m)\, \varphi(x, m) = \frac{x}{4} \int_0^1 (1-w)^{-\frac{1}{2}}\, dw \int_0^\infty e^{-u - \frac{x^2 w}{4u}}\, u^{m-2}\, du.$$

Comparing this equation with the equality obtained before

$$\Gamma(m) f(x, m) = \frac{1}{2} \sqrt{\pi} \int_0^\infty e^{-u-\frac{x^2}{4u}} u^{m-\frac{3}{2}} du,$$

it follows, that we may write

$$\Gamma(m) \varphi(x, m) = \frac{x}{2\sqrt{\pi}} \int_0^1 (1-w)^{-\frac{1}{2}} dw \, \Gamma(m - \tfrac{1}{2}) f\left(x\sqrt{w}, m - \frac{1}{2}\right) \text{ [1]}.$$

We now expand $\Gamma(m - \frac{1}{2}) f\left(x \sqrt{w}, m - \frac{1}{2}\right)$ and make the substitution

$$\Gamma(m - \tfrac{1}{2}) f\left(x \sqrt{w}, m - \frac{1}{2}\right) = \frac{\pi}{2 \sin \pi m} \cdot \Gamma(\tfrac{1}{2}) \left\{ - L\left(x \sqrt{w}, m - \frac{1}{2}\right) + \right.$$
$$\left. + \left(\frac{x}{2}\right)^{2m-2} M\left(x \sqrt{w}, m - \frac{1}{2}\right) \right\}.$$

Then integrating with respect to w, we find the desired expansion of $\varphi(x, m)$ in the form

$$\varphi(x, m) = \int_0^\infty \frac{\sin xt}{(1+t^2)^m} dt = \frac{\pi}{2 \sin \pi m} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(m)} \left\{ -N(x, m) + \left(\frac{x}{2}\right)^{2m-1} M(x, m) \right\},$$

where $N(x, m)$ represents the new series

$$N(x, m) = \sum_{h=0}^{h=\infty} \frac{\left(\frac{x}{2}\right)^{2h+1}}{\Gamma(h + \frac{1}{2}) \, \Gamma(-m + 2 + h)}.$$

The same remarks as were made concerning the first integral $f(x, m)$, can here be made again. The integral has only a meaning for real values of x and for positive values of m, but from the expansion is inferred, that the integral incompletely represents a function of x which exists over the whole x-plane, quite independently of the values assigned to the parameter m. Again the origin $x = 0$ and $x = \infty$ are the only singularities of the function. The singularities are logarithmic, when m is an integer and the origin becomes a regular point, when $2m$ is equal to an odd integer

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[1] It is possible to invert this relation. It may be shewn that we have also

$$\Gamma(m) f(x, m) - \frac{1}{2} \sqrt{\pi} \, \Gamma(m - \tfrac{1}{2}) =$$
$$= - \frac{x}{2\sqrt{\pi}} \int_0^1 (1-w)^{-\frac{1}{2}} dw \, \Gamma(m - \tfrac{1}{2}) \varphi\left(x\sqrt{w}, m - \frac{1}{2}\right).$$

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$2k+1$, but in no case is a finite expression by means of elementary functions obtainable.

The function $\varphi(x, m)$ as well as $f(x, m)$ satisfies the relation

$$D^h_{\alpha=1} \left\{ \alpha^{-m+\frac{1}{2}} \varphi(x \sqrt{\alpha}, m) \right\} = \frac{(-1)^h \, \Gamma(m+h)}{\Gamma(m)} \varphi(x, m+h),$$

and by means of this rule expansions for $\varphi(x,k)$ and $\varphi\left(x, k+\frac{1}{2}\right)$ may be deduced from the equations

$$\varphi(x,1) = \int_0^\infty \frac{\sin xt}{1+t^2} dt = \sum_{h=0}^{h=\infty} \frac{x^{2h+1}}{(2h+1)!} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{2h+1} - C - \log x\right) =$$

$$= \frac{1}{2} \left\{ e^{-x} Li(e^{+x}) - e^{+x} Li(e^{-x}) \right\},$$

and

$$\varphi\left(x, \frac{1}{2}\right) = \int_0^\infty \frac{\sin xt}{\sqrt{1+t^2}} dt = \frac{\pi}{2} \sum_{h=0}^{h=\infty} \frac{\left(-\frac{x}{2}\right)^h}{\Gamma\left(\frac{h}{2}+1\right)^2}.$$

Botany. — On "Leptostroma austriacum OUD., *a hitherto unknown Leptostromacea living on the needles of Pinus austriaca; and on* Hymenopsis Typhae (Fuck.) SACC., *a hitherto insufficiently described Tuberculariacea, occurring on the withered leafsheaths of Typha latifolia.*" By Prof. C. A. J. A. OUDEMANS.

1. LEPTOSTROMA AUSTRIACUM OUD.

(Plate I.)

On the 13th of June 1904 I received from Dr. J. RITZEMA BOS, Professor at Amsterdam, a number of specimens of transplanted seedlings of *Pinus austriaca*, originating from Schoorl, all dead and of which the accompanying letter informed me that the roots showed here and there cushionlike prominences, the surface of which was covered with shuttle-shaped conidia, divided into cells, and the microscopic properties of which resembled most those of conidia of the genus *Fusarium*.

Besides I found, without my attention having been directed to it, that most needles of the dead plantlets were spotted on both