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be necessary, however, at least for low temperatures, to take great care that the temperature is kept constant during the time required for a measurement. In these measurements the determination of the temperature was less accurate than seemed desirable with a view to the accuracy of the determination of the ratio during the measurements on one day.

In the results there is a striking difference between the gold and the platinum. Though the values found do not help us to fix the temperature function for gold for want of certainty about the zero to which they belong, yet they show that the curvature of the line which for gold represents that temperature function is much smaller than in the case of platinum and that the curve is bent more towards the absolute zero. Hence a gold wire would be more suited for extrapolation than a platinum wire, because here the deviations which we cannot but expect, are much smaller.

Mathematics. — "*A congruence of order two and class two formed by conics*". By Prof. J. DE VRIES.

For a twofold infinite system of conics (congruence) *order* is called the number of conics through an arbitrary point, *class* the number of conics with an arbitrary right line for bisecant.

The congruences of order one and class one arise from the projective coordination of a net of planes to a net of quadrics[1]). Under investigation were furthermore the congruences of order one and class two and those, the conics of which cut a fixed conic twice[2]).

In this communication the characteristic numbers are deduced of the congruence determined by the tangent planes of a quadric Q^2 on the planes of a net $[Q^2]$ of quadrics to which they are projectively conjugate.

2. To obtain this conjugation we project the points P of Q^2 out of a fixed point P_0 of Q^2 on a plane Φ. A projectivity between the points P' of Φ and the surfaces of $[Q^2]$ furnishes then immediately a projectivity between $[Q^2]$ and the system $[\pi]_2$ of the tangent planes π of Q^2.

To a pencil (Q^2) in $[Q^2]$ corresponds a range of points (P') in

[1]) D. MONTESANO, Su di un sistema lineare di coniche nello spazio, Atti di Torino, 1891—1892, t. XXVII, p. 660.

[2]) M. PIERI, Sopra alcune congruenze di coniche, Atti di Torino, 1892—1893, t. XXVIII, p. 135.

Φ, thus a conic on Q^2, thus the system of the tangent planes π passing through a fixed point T. Through T and a point X of the base-curve of (Q^2) two planes π pass; wherefore X bears two conics of the congruence, which is thus of *order two* $(P = 2)$.

3. To the tangent planes π through an arbitrary point T correspond the points P of a conic not passing through P_0, having thus for image a conic in Φ. So to this system $(\pi)_2$, of index two, is conjugate a system $(Q^2)_2$ possessing likewise index two, having two surfaces in common with each pencil (Q^2). When considering the ranges of points determined by the projective systems $(\pi)_2$ and $(Q^2)_2$ on an arbitrary right line we find that they generate a surface T^6 of degree six, which is the locus of the conics of the congruence the planes of which pass through a fixed point T. Hence we get $\mu\,\nu = 6$.

4. Through two arbitrary points pass two tangent planes π, hence the planes of two conics; so an arbitrary right line is bisecant of two conics, and the congruence is of *class two* $(\mu^2 = 2)$.

The numbers $P = 2$, $\mu\,\nu = 6$ and $\mu^2 = 2$ satisfy the well known formula $P = \mu\,\nu - 2\,\mu^2$.

Through a right line of Q^2 pass an infinite number of planes π; the conics they bear form a cubic surface.

As each ray through T meets two conics, T^6 has in T a double point. If A^2 is one of the conics on which T is situated, T^6 is touched in T by each bisecant of A^2 out of T. So T is a biplanar point.

If T is one of the eight base points of the net $[Q^2]$ then T^6 has in T a fourfold point; for on every ray through T lie but two points besides T.

5. Let us take for Q^2 the paraboloid $x\,y = z$, then the substitution $x = \alpha\,\varrho$, $y = \beta\,\varrho$, $z = \gamma\,\varrho$ furnishes first $\varrho = \gamma : \alpha\,\beta$ and then

$$x = \gamma : \beta \quad , \quad y = \gamma : \alpha \quad , \quad z = \gamma^2 : \alpha\,\beta .$$

So the tangent planes π are represented by

$$\beta\,\gamma\,x + \alpha\,\gamma\,y - \alpha\,\beta\,z - \gamma^2 = 0.$$

The above-indicated conjugation is arrived at by putting

$$\alpha\,\mathrm{A} + \beta\,\mathrm{B} + \gamma\,\mathrm{C} = 0,$$

where A, B, C are quadratic functions of x, y, z. We represent their coefficients by a_{kl}, b_{kl}, c_{kl} and we write briefly

$$d_{kl} = \alpha\,a_{ik} + \beta\,b_{ik} + \gamma\,c_{ik}.$$

If a Q^2 of the net is to be touched by the conjugate plane π, then

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} & \beta\gamma \\ d_{12} & d_{22} & d_{23} & d_{24} & \alpha\gamma \\ d_{13} & d_{23} & d_{33} & d_{34} & -\alpha\beta \\ d_{14} & d_{24} & d_{34} & d_{44} & -\gamma^2 \\ \beta\gamma & \alpha\gamma & -\alpha\beta & -\gamma^2 & 0 \end{vmatrix} = 0.$$

must be satisfied.

We find here a relation

$$D_7(\alpha, \beta, \gamma) = 0,$$

which is homogeneous and of degree 7 in α, β, γ. If we regard these parameters as homogeneous coordinates, this relation represents a curve of degree 7 possessing nodes in the points $A\,(\beta = 0, \gamma = 0)$ and $B\,(\alpha = 0, \gamma = 0)$.

6. For the conics passing through point $T(x_1, y_1, z_1)$ we have the relation

$$M_2(\alpha, \beta, \gamma) \equiv x_1\,\beta\gamma + y_1\,\alpha\gamma - z_1\,\alpha\beta - \gamma^2 = 0.$$

It is represented by a conic passing through A and B.

Besides A and B the auxiliar curves D^7 and M^2 have ten points in common. So through T pass the planes of ten conics each degenerated into two right lines ($\delta\mu = 10$).

That the points A and B must not be taken into consideration is shown as follows: For $\alpha = 0, \gamma = 0$ we find $\mathrm{B} = 0$ and $y = 0 : 0$, thus the pencil of planes around OX; of these tangent planes of course only one is conjugate to $\mathrm{B} = 0$ and the conic determined by it does not form a pair of lines generally.

Out of the relation [1])

$$3\mu\nu = 2\eta\mu + \delta\mu + 4\mu^2$$

ensues, as $\mu\nu = 6$, $\delta\mu = 10$ and $\mu^2 = 2$,

$$\eta\mu = 0.$$

This could be foreseen, for the cones of $[Q^2]$ form a system ∞^1; the number of those cones touched by the homologous planes π is thus finite and all twofold symbols in which η appears have therefore the value zero.

7. The right line $x = 0$, $y = 0$ is cut by the conics for which we have

$$\alpha\beta z + \gamma^2 = 0 \quad \text{and} \quad d_{33}\,z^2 + 2d_{34}\,z + d_{44} = 0,$$

1) Compare my communication in these Proceedings, p. 264.

thus

$$N_5(\alpha, \beta, \gamma) \equiv d_{33}\gamma^4 - 2d_{34}\alpha\beta\gamma^2 + d_{44}\alpha^2\beta^2 = 0.$$

The curve N^5 representing this relation has evidently nodes in A and B.

By connecting N^5 with M^2 and D^7 we find anew $\mu\nu = 6$ and farther

$$\delta\nu = 27.$$

The pairs of lines of the congruence form a skew surface of degree 27.

8. To find the characteristic numbers containing the symbol ϱ we consider the pairs of points which the conics of the congruence have in common with the plane $z = 1$. They are indicated by

$$\beta\gamma x + \alpha\gamma y = \alpha\beta + \gamma^2,$$

$$d_{11}x^2 + 2d_{12}xy + d_{22}y^2 + 2(d_{13} + d_{14})x + 2(d_{23} + d_{24})y + (d_{33} + 2d_{34} + d_{44}) = 0.$$

So for the conics touching $z = 1$

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} + d_{14} & \beta\gamma \\ d_{12} & d_{22} & d_{23} + d_{24} & \alpha\gamma \\ d_{13} + d_{14} & d_{23} + d_{24} & d_{33} + 2d_{34} + d_{44} & -\alpha\beta - \gamma^2 \\ \beta\gamma & \alpha\gamma & -\alpha\beta - \gamma^2 & 0 \end{vmatrix} = 0.$$

This is a relation

$$R_6(\alpha, \beta, \gamma) = 0,$$

which is represented by a curve R^6 having A and B for nodes.

By combining R^6 and D^7, M^2 and N^5 we find successively

$$\delta\varrho = 34, \qquad \mu\varrho = 8, \qquad \nu\varrho = 22.$$

From this ensues that the skew surface of the pairs of lines has a double curve of degree 17 and that the conics touching a given plane (in particular thus the parabolae of the congruence) form a surface of degree 22.

Out of the relations

$$3\nu^2 = \delta\nu + 4\mu\nu \quad \text{and} \quad 3\varrho^2 = 2\delta\varrho + 2\mu\varrho$$

we finally find for the missing characteristic numbers

$$\nu^2 = 17 \quad \text{and} \quad \varrho^2 = 28.$$

So the conics cutting a fixed right line form a surface of order 17.