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containing only $x_1, x_2, \dots x_p$ and thus expressing that the projection Q of P on $O(X_1\, X_2 \dots X_p)$ remains in this latter space on the quadratic space represented by (1), then the relation holds good

$$tg\ a = \frac{1}{OR}.$$

If $b_1, b_2, \dots b_p$ are the half axes of this new quadratic space, we shall find

$$tg\ a_1 = \frac{1}{b}, tg\ a_2 = \frac{1}{b_2}, \dots, tg a_p = \frac{1}{b_p}.$$

Now (1) passes into the symbolic form

$$(A_1\, x_1 + A_2\, x_2 + \dots + A_p\, x_p)^{(2)} = 1$$

by the substitutions

$$\sum_{i=1}^{n-p} a^2{}_{k,i} = A_{k,k} \quad , \quad \sum_{i=1}^{n-p} a_{k,i}\, a_{l,i} = A_{k,l};$$

so the well know secular equation

$$\begin{vmatrix} A_{11} - \lambda & A_{21} & \dots & A_{p,1} \\ A_{12} & A_{22} - \lambda & \dots & A_{p,2} \\ \dots & \dots & \dots & \dots \\ A_{1,p} & A_{2,p} & & A_{p,p} - \lambda \end{vmatrix} = 0$$

furnishes by its roots $\lambda_1, \lambda_2, \dots \lambda_p$ the coefficients of the equation of that quadratic space reduced on the axes.

From the relations

$$\lambda_1 = \frac{1}{b_1{}^2}, \lambda_2 = \frac{1}{b_2{}^2}, \dots \lambda_p = \frac{1}{b_p{}^2}$$

ensues immediately that the demanded equation is arrived at by replacing in the above mentioned determinant λ by $tg^2\, a$.

Mathematics. — *"The locus of the principal axes of a pencil of quadratic surfaces"*. By Prof. J. CARDINAAL.

1. The envelope of the axes of a pencil of conics was investigated among others by M. TREBITSCHER[1]). He found that the axes of the above mentioned conics envelop a curve of class three touching the right line at infinity of the plane in two points conjugate to the directions of the axes of the two parabolae of the pencil with respect to the

[1]) Ueber Beziehungen zwischen Kegelschnittbüscheln und rationalen Curven dritter Classe, Sitzungsber. der Kaiserl. Akademie der Wissenschaften, Bnd. LXXXI, p. 1080.

isotropic points I and J. So the curve is of order four, i.e. rational. This result is mentioned, in the "Encyklopädie der Mathematischen Wissenschaften" III, p. 101. However, if we consult in the same work the theory of the quadratic surfaces we find no evidence of an attempt to determine the locus of the principal axes of the surfaces of a pencil. The present writer makes it his aim in the following to publish some investigations on this locus.

2. We presuppose a simpler special case of the pencil and we take a pencil of concentric quadratic cones, of which the locus of the principal axes is a cone the order of which can be determined. Let us suppose to this end the section of one of the cones with the plane at infinity; the conic formed in this way determines with the isotropic circle a common autopolar triangle and the vertices of that triangle determine the directions of the principal axes of the cone. From this follows:

The principal axes of all the cones of the pencil cut the plane at infinity in the vertices of the common autopolar triangles of the conics situated in this plane and the isotropic circle. These triplets of points form the Jacobian curve of the net of conics determined by two of the conics and the isotropic circle.

So the cone of the axes is a cone of order three cutting the plane at infinity in the just mentioned Jacobian curve.

To realize the position of the principal axes of this cubic cone we choose a generatrix a_1. If we assume a plane through the vertex normal to a_1 this will cut the cone according to three rays a_2, a_3, b_1; a_2 and a_3 are normal to each other, b_1 belongs to an other trieder of axes, obtained by assuming through the vertex a plane normal to b_1; this plane passes through a_1 and cuts the cone moreover in the two principal axes b_2 and b_3 normal to each other.

As a rule this cone will not have a nodal generatrix, so it will not be rational.

3. Suppose a pencil of quadratic surfaces be given. Out of a point O in space as vertex we construct the parallel cones of the asymptotic cones of the various surfaces; in this manner a pencil of cones is formed, with respect to which we can construct the cone of the axes. The trieders of axes of this cone are parallel to the trieders of axes of the surfaces of the pencil.

Let further a skew cubic φ_3 be constructed, which is the locus of the centres of the surfaces of the pencil; if then out of each centre a trieder is constructed parallel to the corresponding trieder of axes of the cone, the surface is formed which is the locus of the principal axes. From this ensues:

The locus of the principal axes of quadratic surfaces belonging to a pencil is a skew surface of which one of the directrix curves is a skew cubic φ_3 possessing a director cone; each point of the skew cubic is homologous to a trieder of rays of the cone.

4. The order of the surface can be determined by investigating by how many principal axes an arbitrary right line l is cut, or how many planes possessing a principal axis can be made to pass through l, which comes to the same thing.

Let A be a point of φ_3, to which three points A'_1, A'_2, A'_3 correspond on the Jacobian curve C_3 in the plane at infinity P_∞. Let moreover P be a plane through l; then this cuts φ_3 in three points A, B, C, to which correspond again in P_∞ the points $A'_1\, A'_2 \ldots C''_3$, so to the plane P correspond through l nine planes $P'_1\, P'_2 \ldots P'_9$.

If reversely we assume a point A' on C_3 only *one* point A on φ_3 corresponds to it. If we now make a plane P' pass through l, it cuts C_3 in three points A', B', C', to which correspond three points A, B, C; so to a plane P' correspond three planes P. From this ensues:

The two coaxial pencils of planes P and P' have a (3,9)-correspondence. So the number of elements of coincidence amounts to 12. From this reasoning, however, we may not conclude that the order of the skew surface is to be 12; this number must be diminished by the number of points common to φ_3 and C_3. The three points of intersection of φ_3 and P_∞ are namely situated on C_3; if we call one of these points S, then S_1 coincides with S quite independently of the position of the assumed right line l. So of the 12 planes of coincidence 3 pass through the points of intersection of φ_3 and C_3; so 9 remains for the order of the skew surface.

5. A full investigation of this surface O_9 is a very extensive one; however, we can consider some properties and trace some particularities. From the plan of the problem ensues that from each point of φ_3 three generatrices can be drawn meeting P_∞ in the three corresponding points; so φ_3 is a threefold curve of O_9.

The section of P_∞ and O_9 possesses some particularities which we shall look into. In the very first place lie on it the three centres S_1, S_2, S_3 of the paraboloids of the pencil supposed to be real for the present. Out of each of those points two principal axes can be drawn having therefore twelve common points of intersection. Moreover each of these axes cuts C_3 in two more points, which can thus be regarded as double points. One of these points belongs however to a triplet of points corresponding to a point of intersection of φ_3 and C_3; so it can be regarded as a point of contact of the plane

P_∞ and O_9. If we combine these results, we arrive at the following theorem:

The section of O_9 and P_∞ is a degenerated curve of order nine broken up into a plane cubic and six right lines. On this section are situated twelve nodes, points of intersection of the principal axes two by two; moreover six nodes are situated on it, formed each time by one of the points of intersection of a principal axis with C_3, and six points of contact, which are the remaining points of intersection. So P_∞ is a sixfold tangent plane of O_8.

So we come to the conclusion that O_9 possesses besides the threefold curve φ_3 still a nodal curve of which for the present we cannot make out how it is composed, but of which the total order is 18.

The number of points of intersection of this curve with one of the generatrices of O_9 can be determined. Let a be one of the right lines connecting a point A of φ_3 with one of the corresponding points A'_1 on C_3. An arbitrary plane Q through a cuts φ_3 in two more points B and C to which correspond on C_3 two triplets of points B'_1, B'_2, B'_3 and C'_1, C'_2, C'_3. In like manner a plane Q' through a cuts the curve C_3 in two more points to which correspond two points on φ_3; so there exists between the pencils of planes Q and Q' a (6,2)-correspondence and the number of planes of coincidence amounts to 8. So all together a is cut by 8 principal axes. As in the general case this number must be diminished by 3, for now too the three points of intersection of φ_3 and C_3 must be taken into account; so a is cut by five principal axes. Each generatrix of O_9 has thus five points in common with the nodal curve.

From the preceding is apparent that the general section of the surface possesses 18 nodes and 3 triple points; if we have in mind that the latter are equivalent to 9 nodes we see that the general section is not rational, as a curve of order nine can have 28 nodes and the curve under investigation possesses only 27 nodes.

6. We shall consider a single case, where the surface O_9 is simplified. We have already noticed that the cone of axes is of order three without nodal generatrix; there would be one if the net of conics possessed in P_∞ a point, common to all conics. As however to this pencil belongs the isotropic circle this case is excluded; it may however happen that the cone of axes breaks up into a quadratic cone and a plane, or into three planes.

7. We choose an example of the first case. When the cone of axes breaks up into a quadratic cone and a plane, then the Jacobian curve in P_∞ must degenerate into a right line C_1 and a conic C_2. This happens:

a. When the conics of the net pass through two fixed points.

b. When the net possesses a double right line.

We restrict ourselves in this communication to the first of these cases; then the base curve of the pencil of surfaces is circular.

It is in the first place necessary now that the cone is degenerated into two parts to consider the distribution of the axes on cone and plane. If the base curve of the pencil of surfaces is circular, there is a system of parallel planes so that each plane is cut according to a pencil of circles. Of each surface of the pencil *one* principal axis runs parallel to these planes. From this ensues:

When in consequence of the existence of a circular base curve the cone of axes degenerates into a quadratic cone and a plane, then of the three points A'_1, A'_2, A'_3, homologous to a point A on φ_3 one lies on the right line C_1 in P_∞ and two on the conic C_2. So the skew surface O_9 degenerates into two other skew surfaces intersecting each other in their common directrix φ_3. For one skew surface φ_3 is a nodal curve, for the other it is single. This already suggests that the former of the two skew surfaces is of order six, the latter of order three. This can be reasoned more minutely in the following manner:

Let l be once more a right line; a plane P through l has three points A, B, C in common with φ_3 to which six points $A'_1, A'_2 \ldots C'_1, C''_2$ on C_2 correspond; so six planes P' correspond to P; if reversely we make a plane P' to lie through l, it cuts C_2 in two points to which on φ_3 two points correspond, so that between the planes P and P' a (2, 6)-correspondence exists. However φ_3 has a point in common with C_1, as C_1 contains the point of contact of a hyperbolic paraboloid of the pencil with P_∞; so there remain for φ_3 two points of contact with C_2 and the order 8, which would arise on account of the (2, 6)-correspondence, must be diminished by 2; so we get a skew surface O_6 of order six. The second skew surface is of order three.

In the general case the section of P_∞ and O_9 consisted, besides of C_3, of three pairs of right lines, to be called a_1a_2, b_1b_2, c_1c_2. If O_9 degenerates in the manner described above these right lines will also be distributed themselves on O_6 and O_3. Let A' again be the point where φ_3 cuts the right line C_1, thus the point of contact of a hyperbolic paraboloid of the pencil; through A' pass the two principal axes a_1a_2 and these belong to O_6, whilst the principal axis not lying in P_∞ through A' belongs to O_3. To O_3 belongs thus one principal axis of each of the pairs b_1b_2 and c_1c_2, so P_∞ is a double tangent plane of O_3 and the section of O_6 and P_∞ consists of the conic C_2, the pair of axes a_1a_2 and the principal axes b_1 and c_1. Of a_1 and a_2 the point of intersection a_1a_2 is the node in the curve of intersection of P_∞ and O_6, one of each of the points of intersection of a_1 and a_2 with C_2' is point of contact; so on a_1 as well as on a_2 another node is situated. Of each of the points of intersection of b_1 and c_1 with C_2 *one* is point of contact, the other is also point of intersection of φ_3 and C_2'. So the points of intersection b_1, c_1, a_1, a_2 mutually are left as nodes; these are five in number besides the point of intersection counted already $a_1 a_2$ belonging to φ_3. So the entire number of the nodes of the section of O_6 and P_∞ not lying on φ_3 amounts to 7. From this ensues that O_6 has besides φ_3, another double curve of order seven. If we make a plane to pass through a generatrix O_1 and if we investigate how many right lines are situated in it, we shall find the number to be 3 corresponding to former results.

The nodal curve of order seven is thus intersected three times by the right lines of O_6.

8. The closer investigation of the surface O_9 as well as that of O_6 and the other possible forms appearing by variously assuming the pencil of surfaces, gives rise to very extensive considerations, which are not to be included in this communication, as for the present its aim was but to show the general properties of the discussed locus.

Physics. — "*Simplified Deduction of the Field and the Forces of an Electron, moving in any given way.*" By Prof. A. Sommerfeld. (Communicated by Prof. H. A. Lorentz).

§ 1. *Summary.*

In the "Göttinger Nachrichten" [1] I communicated a general representation of the field of an electron, moving in any given way, which seems to be simpler than the formulae, hitherto known, which are based on the work of H. A. Lorentz. This is the difference: My formulae express the potentials by a *simple integral, extending over the past time* and containing only the varying distances of the point in question *from the centre of the electron,* supposed to be spherical, whereas the formulae hitherto known are *double or triple integrals,* extending over the *space,* charged with electricity, and containing the distance of the point in question *from the position of the charge at a certain former time.* It may be remarked, that

1) Nachrichten d. K. Gesellschaft d. Wissenschaften 1904 Heft 2; in the following to be cited as "first paper".