Mathematics. — *The values of some definite integrals connected with Bessel functions*". By Dr. W. KAPTEYN.

The integrals referred to are

$$P = \int_0^{2\pi} \frac{\cos(x \sin\theta) - \cos(x \sin\varphi)}{\cos\theta + \cos\varphi} \, d\theta,$$

$$Q = \int_0^{2\pi} \frac{\sin(x \sin\theta)\sin\theta - \sin(x \sin\varphi)\sin\varphi}{\cos\theta + \cos\varphi} \, d\theta,$$

$$R = \int_0^{2\pi} \frac{\cos(x \cos\theta) - \cos(x \cos\varphi)}{\cos\theta + \cos\varphi} \, d\theta,$$

$$S = \int_0^{2\pi} \frac{\sin(x \cos\theta)\cos\theta - \sin(x \cos\varphi)\cos\varphi}{\cos + \cos\varphi} \, d\theta.$$

If in these integrals we insert the wellknown equations

$$\cos(x \sin\theta) = I_0 + 2 I_2 \cos 2\theta + 2 I_4 \cos 4\theta + \ldots$$
$$\sin(x \sin\theta)\sin\theta = I_1 (1 - \cos 2\theta) + I_3 (\cos 2\theta - \cos 4\theta) + \ldots$$
$$\cos(x \cos\theta) = I_0 - 2 I_2 \cos 2\theta + 2 I_4 \cos 4\theta - \ldots$$
$$\sin(x \cos\theta)\cos\theta = I_1 (1 + \cos 2\theta) - I_3 (\cos 2\theta + \cos 4\theta) + \ldots$$

where I_p stands for the Bessel function $I_p(x)$ of order p, and if we write

$$A_{2n} = \int_0^{2\pi} \frac{\cos 2n\theta - \cos 2n\varphi}{\cos\theta + \cos\varphi} \, d\varphi$$

it is easy to find

$$P = I_0 A_0 + 2 I_2 A_2 + 2 I_4 A_4 + \ldots$$
$$Q = I_1 A_0 + (I_3 - I_1) A_2 + (I_5 - I_3) A_4 + \ldots$$
$$R = -2 I_2 A_2 + 2 I_4 A_4 - 2 I_6 A_6 + \ldots$$
$$S = I_1 A_2 - I_3 (A_2 + A_4) + I_5 (A_4 + A_6) - \ldots$$

In order to determine A_{2n} I notice that

$$(a) \quad \sin\varphi \frac{\cos 2n\theta - \cos 2n\varphi}{\cos\theta + \cos\varphi} = -\sin 2n\varphi + 2 \sin(2n-1)\varphi \cos\theta - $$
$$- 2 \sin(2n-2)\varphi \cos 2\theta + \ldots$$
$$+ 2 \sin\varphi \cos(2n-1)\theta$$

This formula can be proved as follows:

If we multiply the 2nd member of this equation by $\cos\theta + \cos\varphi$ we find

2nd member
$$\times \cos\theta = \sin(2n-1)\varphi + \sum_{p=1}^{2n-1} (-1)^p [\sin(2n-p+1)\varphi + \sin(2n-p-1)\varphi] \cos p\theta + $$
$$+ \sin\varphi \cos 2n\theta$$
$$\times \cos\varphi = -\sin 2n\varphi \cos\varphi + \sum_{p=1}^{2n-1} (-1)^{p-1} [\sin(2n-p+1)\varphi + \sin(2n-p-1)\varphi] \cos p\theta;$$

so the sum becomes

$$\sin(2n-1)\varphi - \sin 2n\varphi \cos\varphi + \sin\varphi \cos 2n\theta = \sin\varphi (\cos 2n\theta - \cos 2n\varphi).$$

From the formula (a) follows immediately

$$A_{2n} = -2\pi \frac{\sin 2n\varphi}{\sin\varphi}.$$

If we replace this value in the expressions just found, we arrive at

$$P = -\frac{4\pi}{\sin\varphi} [I_2 \sin 2\varphi + I_4 \sin 4\varphi + I_6 \sin 6\varphi + \ldots],$$

$$Q = -\frac{2\pi}{\sin\varphi} [(I_3 - I_1) \sin 2\varphi + (I_5 - I_3) \sin 4\varphi + \ldots]$$

$$= 4\pi [I_1 \cos\varphi + I_3 \cos 3\varphi + I_5 \cos 5\varphi \ldots],$$

$$R = \frac{4\pi}{\sin\varphi} [I_2 \sin 2\varphi - I_4 \sin 4\varphi + I_6 \sin 6\varphi - \ldots],$$

$$S = -\frac{2\pi}{\sin\varphi} [I_1 \sin 2\varphi - I_3 (\sin 2\varphi + \sin 4\varphi) + I_5 (\sin 4\varphi + \sin 6\varphi) \ldots],$$

$$= -\frac{4\pi \cos\varphi}{\sin\varphi} [I_1 \sin\varphi - I_3 \sin 3\varphi + I_5 \sin 5\varphi - \ldots].$$

Moreover from the formula (a) we can deduce another result: When we develop

$$\frac{\cos 2n\theta - \cos 2n\varphi}{\cos\theta + \cos\varphi} = \tfrac{1}{2} a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots$$

then we know that

$$a_p = \frac{2}{\pi} \int_0^\pi \frac{\cos 2n\theta - \cos 2n\varphi}{\cos\theta + \cos\varphi} \cos p\theta \, d\theta$$

If we compare this to the equation (a) we arrive at

$$a_p = (-1)^{p-1}\, 2\, \frac{\sin(2n-p)\varphi}{\sin\varphi}$$

for $p = 0, 1, 2, \ldots, (2n-1)$, whilst for greater values of p we have $a_p = 0$.