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$D_1$  are most readily interpreted by considering them as originating from a triplet and not from a sextet.

It seems rather superfluous to give any further explanation of figs. 8, 4, 5; in the case relating to fig. 5, the vapour density is again greater than in fig. 4. All the phenomena we have considered are qualitatively in excellent accordance with VOIGT's theory.

The phenomena described for  $D_1$  and  $D_2$  again demonstrate the existence of very characteristic differences between different spectral lines, differences no less striking here than in the case of the related phenomena of the magnetic separation of the spectral lines and of the rotation of the plane of polarization in the interior<sup>1)</sup> of, and close to, the absorption line. It is certainly very interesting that the theory explains the entirely different behaviour of  $D_1$  and  $D_2$  in the case now considered by differences between the velocities of propagation of vibrations normal and parallel to the field, assuming, of course, the magnetic division of the lines.

**Physics.** — "*The motion of electrons in metallic bodies*". I. By Prof. H. A. LORENTZ.

It has been shown by RIECKE<sup>2)</sup>, DRUDE<sup>3)</sup> and J. J. THOMSON<sup>4)</sup> that the conductivity of metals for electricity and heat, the thermoelectric currents, the THOMSON-effect, the HALL-effect and phenomena connected with these may be explained on the hypothesis that a metal contains a very large number of free electrons and that these particles, taking part in the heat-motion of the body, move to and fro with a speed depending on the temperature. In this paper the problems to which we are led in theories on these subjects will be treated in a way somewhat different from the methods that have been used by the above physicists.

§ 1. I shall begin by assuming that the metal contains but one

<sup>1)</sup> ZEEMAN, Proc. Acad. Amsterdam May 1902, see also the description of another phenomenon in VOIGT, Gottinger Nachrichten, Heft 5, 1902.

<sup>2)</sup> E. RIECKE, Zur Theorie des Galvanismus und der Wärme, Ann. Phys. Chem. 66 (1898), p. 353, 545. 1199; Ueber das Verhalten der Leitfähigkeiten der Metalle für Wärme und für Elektrizität, Ann. Phys. 2 (1900), p. 835.

<sup>3)</sup> P. DRUDE, Zur Elektronentheorie der Metalle, Ann. Phys. 1 (1900), p. 566; 3 (1900), p. 369.

<sup>4)</sup> J. J. THOMSON, Indications relatives à la constitution de la matière fournies par les recherches récentes sur le passage de l'électricité à travers les gaz, Rapports du Congrès de physique de 1900, Paris, 3, p. 138

*Triplet*

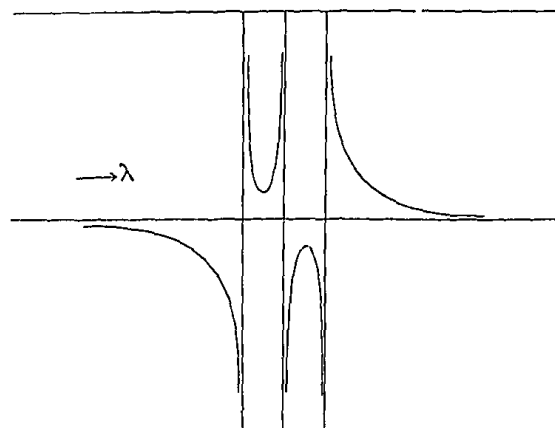


Fig 1  
(theoretical curve)



Fig 4  
(observed)



Fig 5  
(observed)

*Quartet*

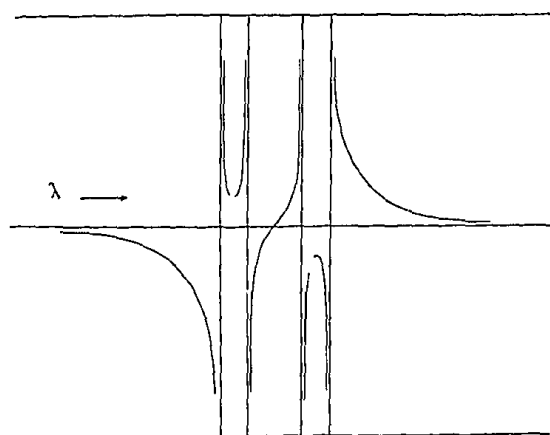


Fig 2  
(theoretical curve)



Fig 6  
(observed)



Fig 7  
(observed)

*Sextet*

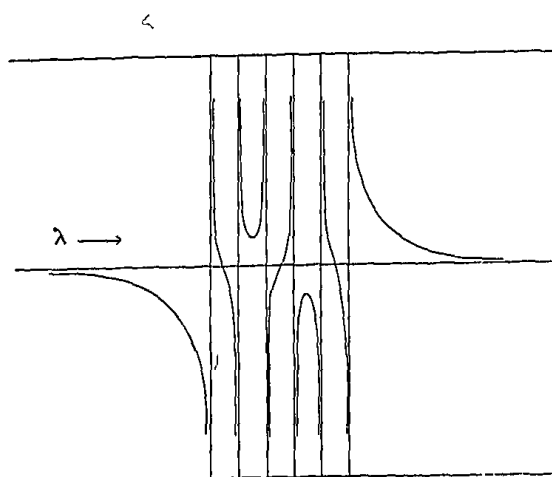


Fig 3  
(theoretical curve)



Fig 8.  
(observed)

kind of free electrons, having all the same charge  $e$  and the same mass  $m$ ; the number of these particles per unit volume will be represented by  $N$ , and I shall suppose their heat-motion to have such velocities that, at a definite temperature, the mean kinetic energy of an electron is equal to that of a molecule of a gas. Denoting by  $T$  the absolute temperature, I shall write for this mean kinetic energy  $\alpha T$ , where  $\alpha$  is a constant.

We shall further consider a cylindrical bar, unequally heated in its different parts, so that, if  $x$  is reckoned along its length,  $T$  is a function of this coordinate. We shall also suppose each electron to be acted on, in the direction of  $OX$ , by a force  $mX$ , whose intensity is a function of  $x$ . Such a force may be due either to an electric field or, in the case of a non-homogeneous metal, to a molecular attraction exerted by the atoms of the metal. Our first purpose will be to calculate the number of electrons  $\nu$  and the amount of energy  $W$  crossing an element of surface perpendicular to the axis of  $x$  in the positive direction, or rather the difference between the numbers of particles in one case and the quantities of energy in the other that travel towards the positive and towards the negative side. Both quantities  $\nu$  and  $W$  will be referred to unit area and unit time.

This problem is very similar to those which occur in the kinetic theory of gases and, just like these, can only be solved in a rigorous way by the statistical method of MAXWELL and BOLTZMANN.

In forming our fundamental equation, we shall not confine ourselves to the cylindric bar, but take a somewhat wider view of the subject. At the same time, we shall introduce a simplification, by which it becomes possible to go further in this theory of a swarm of electrons than in that of a system of molecules. It relates to the encounters experienced by the particles and limiting the lengths of their free paths. Of course, in the theory of gases we have to do with the *mutual* encounters between the molecules. In the present case, on the contrary, we shall suppose the collisions with the metallic atoms to preponderate; the number of these encounters will be taken so far to exceed that of the collisions between electrons mutually, that these latter may be altogether neglected. Moreover, in calculating the effect of an impact, we shall treat both the atoms and the electrons as perfectly rigid elastic spheres, and we shall suppose the atoms to be immovable. Of course, these assumptions depart more or less from reality; I believe however that we may safely assume the general character of the phenomena not to be affected by them.

§ 2. Let  $dS$  be an element of volume at the point  $(x, y, z)$ . At

the time  $t$ , this element will contain a certain number (in fact, a very large number) of electrons moving in different ways.

Now, we can always imagine a piece of metal of finite dimensions, say of unit volume, in which the "concentration", as we may call it, of the electrons and the distribution of the different velocities among them are exactly the same as in the element  $dS$ . In studying the said distribution for the  $N$  electrons, with which we are then concerned, we shall find a diagram representing their velocities to be very useful. This is got by drawing, from a fixed point  $O$ ,  $N$  vectors, agreeing in direction and magnitude with the velocities of the electrons. The ends of these vectors may be called the velocity-points of the electrons and if, through the point  $O$  of the diagram, we draw axes parallel to those used in the metal itself, the coordinates of a velocity-point will be equal to the components  $\xi$ ,  $\eta$ ,  $\zeta$  of the velocity of the corresponding electron.

Writing now

$$f(\xi, \eta, \zeta) d\lambda$$

for the number of velocity-points within the element  $d\lambda$  at the point  $(\xi, \eta, \zeta)$ , we make the exact solution of all problems relating to the system of electrons depend on the determination of the function  $f(\xi, \eta, \zeta)$ .

We may also say that

$$f(\xi, \eta, \zeta) dS d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the number of electrons in the element  $dS$ , whose velocity-points lie in  $d\lambda$ ; in particular

$$f(\xi, \eta, \zeta) dS d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the number of electrons for which the values of the components of velocity are included between  $\xi$  and  $\xi + d\xi$ ,  $\eta$  and  $\eta + d\eta$ ,  $\zeta$  and  $\zeta + d\zeta$ . The expression (2) is got from (1) by a proper choice of the element  $d\lambda$ .

If the function in (1) were known, we could deduce from it the total number of electrons and the quantities  $\nu$  and  $W$  mentioned in § 1. Integrating over the full extent of the diagram of velocities, we have

$$N = \int f(\xi, \eta, \zeta) d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\nu = \int \xi f(\xi, \eta, \zeta) d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and if, in treating of the flux of energy, we confine ourselves to the kinetic energy of the particles,

$$W = \frac{1}{2} m \int \xi r^2 f(\xi, \eta, \zeta) d\lambda \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the latter formula,  $r$  denotes the magnitude of the velocity.

It ought to be observed that, in general, the state of the metal will change from point to point and from one instant to another. If such be the case, the function  $f(\xi, \eta, \zeta)$  will depend on  $x, y, z$  and  $t$ , so that the symbol may be replaced by  $f(\xi, \eta, \zeta, x, y, z, t)$ . We shall, however, often abbreviate it to  $f$ .

As to the integrations in (3), (4) and (5), in performing these, we must treat  $x, y, z$  and  $t$  as constants.

§ 3. We shall now seek an equation proper for the determination of the function  $f$ . For this purpose we fix our attention on the electrons present, at the time  $t$ , in the element  $dS$  at the point  $(x, y, z)$ , and having their velocity-points within the element  $d\lambda$ ; we shall follow these particles, the number of which is

$$f(\xi, \eta, \zeta, x, y, z, t) dS d\lambda \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in their course during the infinitely short time  $dt$ . At the end of this interval those particles of the group which have escaped a collision with an atom will be found in an element  $dS'$ , which we may get by shifting  $dS$  in the directions of the axes over the distances  $\xi dt, \eta dt, \zeta dt$ . At the same time, if there are external forces, the velocities will have changed. I shall suppose each electron to be acted on by the same force  $(mX, mY, mZ)$ . Then, for each of them, the components of the velocity will have increased by  $Xdt, Ydt, Zdt$  and, at the end of the interval  $dt$ , the velocity-points will be found in the element  $d\lambda'$ , which may be considered as the original element  $d\lambda$ , displaced over those distances.

We must further keep in mind that, while travelling from  $dS$  to  $dS'$ , the group (6) loses a certain number of electrons and gains others. Indeed, all particles of the group that strike against an atom have their velocities changed, so that they do not any longer belong to the group, and, on the other hand, there are a certain number of encounters by which electrons having initially different velocities, are made to move in such a way, that their velocity-points lie within  $d\lambda$ . Writing

$$a dS d\lambda dt$$

for the number of electrons leaving the group and

$$b dS d\lambda dt$$

for the number entering it, we may say:

If, to the number (6), we add  $(b - a) dS d\lambda dt$ , we shall get

the number of electrons which, at the time  $t + dt$ , satisfy the conditions that they themselves shall be found in the element  $dS'$  at the point  $(x + \xi dt, y + \eta dt, z + \zeta dt)$  and their velocity-points in the element  $d\lambda'$  at the point  $(\xi + X dt, \eta + Y dt, \zeta + Z dt)$ . Hence, since  $dS' = dS$  and  $d\lambda' = d\lambda$ ,

$$f(\xi, \eta, \zeta, x, y, z, t) + (b - a) dt = \\ = f(\xi + X dt, \eta + Y dt, \zeta + Z dt, x + \xi dt, y + \eta dt, z + \zeta dt, t + dt),$$

or

$$b - a = \frac{\partial f}{\partial \xi} X + \frac{\partial f}{\partial \eta} Y + \frac{\partial f}{\partial \zeta} Z + \frac{\partial f}{\partial x} \xi + \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial z} \zeta + \frac{\partial f}{\partial t}. \quad (7)$$

This is the equation we wanted to establish<sup>1)</sup>.

It is easily seen that, in calculating the numbers of collisions  $a dS d\lambda dt$  and  $b dS d\lambda dt$ , we need not trouble ourselves about the state of the metal varying from one point to another; we may therefore understand by  $a d\lambda dt$  the decrease, and by  $b d\lambda dt$  the increase which the group of electrons characterized by  $d\lambda$  would undergo, if we had to do with a piece of metal occupying a unit of volume and being, in all its parts, in the state that exists in the element  $dS$ .

§ 4. We are now prepared to calculate the values of  $a$  and  $b$ . Let  $R$  be the sum of the radii of an atom and an electron,  $n$  the number of atoms in unit space, and let us in the first place confine ourselves to encounters of a definite kind. I shall suppose that in these the line joining the centres falls within a cone of the infinitely small solid angle  $d\omega$ .

Taking as axis of this cone one of the straight lines that may be drawn in it, and denoting by  $\vartheta$  the *acute* angle between the axis and the direction of motion of the group (6), I find for the number of electrons in this group undergoing an encounter of the kind chosen,

$$n R^2 f(\xi, \eta, \zeta) r \cos \vartheta d\lambda d\omega \quad . \quad . \quad . \quad . \quad . \quad (8)$$

per unit time, a result which leads to the value

$$a = n \pi R^2 f(\xi, \eta, \zeta) r \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if we take into account *all* encounters, whatever be the direction of the line joining the centres.

Now, if we ascribe to a metallic atom so large a mass, that it is not sensibly put in motion by an electron flying against it, the velocity of the latter after the encounter is given by a very simple rule. We have only to decompose the initial velocity into one

<sup>1)</sup> See LORENTZ, Les équations du mouvement des gaz et la propagation du son suivant la théorie cinétique des gaz, Arch. néerl. 16, p. 9.

component along the line of the centres and another perpendicular to it; the latter of these components will remain unchanged and the former will have its direction reversed.

In applying this to the encounters of the particular kind specified at the beginning of this §, we may take for all of them the line of centres to coincide with the axis of the cone  $d\omega$ . Our conclusion may therefore be expressed as follows: Let  $V$  be a plane through the origin in the diagram of velocities, perpendicular to the axis of the cone. Then, the velocity-point of the electron after impact will be the geometrical image of the original point with respect to this plane. It is thus seen that all electrons whose velocity-points before the encounters are found in the element  $d\lambda$  will afterwards have their representative points in  $d\lambda'$ , the image of  $d\lambda$  with respect to the plane  $V$ .

By this it becomes also clear, in what way the number  $b$  can be calculated; indeed, in encounters taking place under the circumstances considered, velocity-points may as well jump from  $d\lambda_r$  to  $d\lambda$  as from  $d\lambda$  to  $d\lambda_r$ . The number of cases in which the first takes place is found from (8), if in this expression we replace  $\xi, \eta, \zeta$  by the coordinates  $\xi', \eta', \zeta'$  of the image of the point  $(\xi, \eta, \zeta)$  with respect to the plane  $V$ . It is to be remarked that the factor  $r \cos \vartheta d\lambda$  may be left unchanged, because the lines drawn from the origin of the diagram to the points  $(\xi, \eta, \zeta)$  and  $(\xi', \eta', \zeta')$  have equal lengths and are equally inclined to the axis of the cone. Also  $d\lambda_r = d\lambda$ . The increase per unit volume of the number of electrons in the group (6), insofar as it is due to encounters in which the line of centres lies within the cone  $d\omega$ , is thus found to be

$$n R^2 f(\xi', \eta', \zeta') r \cos \vartheta d\lambda d\omega$$

and, in order to find  $b$ , it remains only to divide this by  $d\lambda$  and to integrate with respect to all cones that have to be taken into account.

Using the formula (8) we may as well calculate directly the difference  $b-a$ . By this the equation (7) becomes

$$\begin{aligned} n R^2 r \int \{ f(\xi', \eta', \zeta') - f(\xi, \eta, \zeta) \} \cos \vartheta d\omega = \\ = \frac{\partial f}{\partial \xi} X + \frac{\partial f}{\partial \eta} Y + \frac{\partial f}{\partial \zeta} Z + \frac{\partial f}{\partial x} \xi + \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial z} \zeta + \frac{\partial f}{\partial t} . . . \quad (10) \end{aligned}$$

We must now express  $\xi', \eta', \zeta'$  in  $\xi, \eta, \zeta$ . Let  $f, g, h$  be the angles between the axes of coordinates and the axis of the cone  $d\omega$ , this last line being taken in such a direction that it makes the *acute* angle  $\vartheta$  with the velocity  $(\xi, \eta, \zeta)$ . Then

$$\xi' = \xi - 2r \cos \vartheta \cos f, \quad \eta' = \eta - 2r \cos \vartheta \cos g, \quad \zeta' = \zeta - 2r \cos \vartheta \cos h, \quad (11)$$



These formulae show that, as we know already, the magnitude of the velocity  $(\xi', \eta', \zeta')$ , which I shall call  $v'$ , is equal to the magnitude  $v$  of the velocity  $(\xi, \eta, \zeta)$ .

As to the integration in (10), it may be understood to extend to the half of a sphere. Indeed, if in the diagram of velocities, we describe a sphere with centre  $O$  and radius 1, and if  $P$  and  $Q$  are the points of this surface, corresponding to the directions  $(\xi, \eta, \zeta)$  and  $(f, g, h)$ , we must give to the point  $Q$  all positions in which its spherical distance from  $P$  is less than  $\frac{1}{2}\pi$ . For  $d\omega$  we may take a surface-element situated at the point  $Q$ .

§ 5. At the time  $t$  and the point  $(x, y, z)$  the metal will have a certain temperature  $T$  and the number  $N$ , the concentration of the swarm of electrons, a definite value.

Now the assumption naturally presents itself, that, if  $T$  and  $N$  had these values continually and in all points, the different velocities would be distributed according to MAXWELL's law

$$f(\xi, \eta, \zeta) = Ae^{-h^2} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Here, the constants  $A$  and  $h$  are related to the number  $N$  and the mean square of velocity  $\overline{v^2}$  in the following way

$$A = N \sqrt{\frac{h^3}{\pi^3}}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\overline{v^2} = \frac{3}{2h}.$$

Since  $\frac{1}{2} m \overline{v^2} = \alpha T$ , the latter relation may also be put in the form

$$h = \frac{3m}{4\alpha T} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It appears from this that the way in which the phenomena depend on the temperature will be known as soon as we have learned in what way they depend on the value of  $h$ .

§ 6. The function  $f$  takes a less simple form if the state of the metal changes from point to point, so that  $A$  and  $h$  are functions of  $x, y, z$ . In this case we shall put

$$f(\xi, \eta, \zeta) = Ae^{-h^2} + \varphi(\xi, \eta, \zeta), \quad . \quad . \quad . \quad . \quad (15)$$

where  $\varphi$  is a function that has yet to be determined by means of the equation (10). We shall take for granted, and it will be confirmed by our result, that the value of  $\varphi(\xi, \eta, \zeta)$  is very small in comparison with that of  $Ae^{-h^2}$ . In virtue of this, we may neglect the terms depending on  $\varphi(\xi, \eta, \zeta)$  in the second member of (10); this

having already a value different from 0, if we put  $f = Ae^{-hr^2}$ . For a stationary state and for the case of the bar mentioned in § 1, the member in question becomes

$$\left( -2hAx + \frac{dA}{dx} - r^2 A \frac{dh}{dx} \right) \xi e^{-hr^2} \dots \dots (16)$$

As to the left hand side of the equation (10), it would become 0, if we were to substitute  $f = Ae^{-hr^2}$ . Here, we must therefore use the complete value (15), the deviation from MAXWELL's law being precisely the means by which this member may be made to become equal to (16).

The occurrence of the factor  $\xi$  in this last expression makes it probable that the same factor will also appear in the function  $\varphi$ . We shall therefore try to satisfy our equation by putting

$$\varphi(\xi, \eta, \zeta) = \xi \chi(r) \dots \dots (17)$$

This leads to

$$f(\xi, \eta, \zeta) = Ae^{-hr^2} + \xi \chi(r)$$

and

$$f(\xi', \eta', \zeta') = Ae^{-hr'^2} + \xi' \chi(r'),$$

consequently, since  $r' = r$ , if we use (11),

$$f(\xi', \eta', \zeta') - f(\xi, \eta, \zeta) = -2r \cos \vartheta \cos f \chi(r),$$

so that the first member of (10) becomes

$$-2nR^2 r^2 \chi(r) \int \cos^2 \vartheta \cos f d\omega \dots \dots (18)$$

Denoting by  $\mu$  the angle between the velocity  $(\xi, \eta, \zeta)$ , i. e. the line  $OP$ , and the axis of  $x$ , and by  $\psi$  the angle between the planes  $QOP$  and  $XOP$ , I find for (18)

$$\begin{aligned} -2nR^2 r^2 \chi(r) \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} \cos^2 \vartheta (\cos \vartheta \cos \mu + \sin \vartheta \sin \mu \cos \psi) \sin \vartheta d\vartheta d\psi = \\ = -\pi n R^2 r^2 \chi(r) \cos \mu = -\pi n R^2 \xi r \chi(r). \end{aligned}$$

If this is equated to (16), the factor  $\xi$  disappears, so that  $\chi(r)$  may really be determined as a function of  $r$ . Finally, putting

$$\frac{1}{\pi n R^2} = l, \dots \dots (19)$$

we draw from (15) and (17)

$$f(\xi, \eta, \zeta) = Ae^{-hr^2} + l \left( 2hAx - \frac{dA}{dx} + r^2 A \frac{dh}{dx} \right) \frac{\xi}{r} e^{-hr^2} \dots (20)$$

I must add that, as is easily deduced from (9), the quantity  $l$  defined by (19) may be called the mean length of the free paths of the electrons, and that, in the equation (20), the terms in  $\frac{dA}{dx}$  and

$\frac{dh}{dx}$  are very small in comparison with  $A e^{-hr^2}$ , provided only the state of the metal differ very little in two points whose mutual distance is  $l$ . This is seen by remarking that the ratios of the terms in question to  $A e^{-hr^2}$  are of the order of magnitude

$$\frac{l \frac{dA}{dx}}{A} \text{ and } l \overline{r^2} \frac{dh}{dx},$$

and that, in the second of these expressions,  $\overline{r^2}$  is of the same order as  $\frac{1}{h}$ .

If the term in (20) which contains  $X$ , is likewise divided by  $A e^{-hr^2}$ , we get

$$2 h l X.$$

Now,  $2 l X$  is the square of the velocity an electron would acquire if, without having an initial motion, it were acted on by the external force  $m X$  over a distance  $l$ . If this velocity is very small as compared with that of the heat-motion, the term in  $X$  in our equation may also be taken to be much smaller than the term  $A e^{-hr^2}$ .

It appears in this way that there are many cases in which, as we have done, the function  $\varphi(\xi, \eta, \zeta)$  may be neglected in the second member of the equation (7).

The above reasoning would not hold however, if, in the case of two metals in contact with one another, there were a real discontinuity at the surface of separation. In order to avoid this difficulty, I shall suppose the bodies to be separated by a layer in which the properties gradually change. I shall further assume that the thickness of this layer is many times larger than the length  $l$ , and that the forces existing in the layer can give to an electron that is initially at rest, a velocity comparable with that of the heat-motion, only if they act over a distance of the same order of magnitude as the thickness. Then, the last terms in (20) are again very small in comparison with the first.

As yet, a theory of the kind here developed cannot show that the values we shall find for certain quantities relating to the contact of two metals (difference of potential and PELTIER-effect) would still hold in the limit, if the thickness of the layer of transition were indefinitely diminished. This may, however, be inferred from thermodynamical considerations.

§ 7. Having found in (20) the law of distribution of the veloci-

ties<sup>1)</sup>, we are in a position to calculate the quantities  $v$  and  $W$  (§ 1) with which we are principally concerned. If the value (20) is substituted in (4) and (5), the term  $A e^{-hr^2}$  leads to an integral containing the factor  $\xi$ ; this integral vanishes, if taken over the full extent of the diagram of velocities. In the remaining integrals the factor  $\xi^2$  occurs; these are easily found, if we replace  $\xi^2$  by  $\frac{1}{3} r^2$ , the element  $d\lambda$  by  $4\pi r^2 dr$ , and if then we integrate from  $r=0$  to  $r=\infty$ . Taking  $r^2=s$  as a new variable, we are led to the integrals

$$\int_0^\infty s e^{-hs} ds, \quad \int_0^\infty s^2 e^{-hs} ds \quad \text{and} \quad \int_0^\infty s^3 e^{-hs} ds,$$

whose values are

$$\frac{1}{h^2}, \quad \frac{2}{h^3} \quad \text{and} \quad \frac{6}{h^4}.$$

Finally, the "stream of electrons" and the flux of heat are given by

$$v = \frac{2}{3} \pi l \left[ \frac{1}{h^2} \left( 2 h A X - \frac{dA}{dx} \right) + 2 \frac{A}{h^3} \frac{dh}{dx} \right]. \quad (21)$$

$$W = \frac{2}{3} \pi m l \left[ \frac{1}{h^3} \left( 2 h A X - \frac{dA}{dx} \right) + 3 \frac{A}{h^4} \frac{dh}{dx} \right]. \quad (22)$$

These are the equations that will be used in all that follows. For the sake of generality, I shall suppose (though, of course, this is not strictly true) that, if only a proper value be assigned to  $l$ , the formulae may still be applied even if we make other assumptions concerning the metallic atoms and their action on the electrons. From this point of view, we may also admit the possibility of different kinds of electrons, if such there are, having unequal mean lengths of free paths, and of, for each kind,  $l$  varying with the temperature.

Provisionally, we shall have to do with only one kind of electrons, reserving the discussion of the more general case for a future communication.

§ 8 From the equation (21) we may in the first place deduce a formula for the *electric conductivity*  $\sigma$  of the metal.

Let a homogeneous bar, which is kept in all its parts at the same temperature, be acted on by an electric force  $E$  in the direction of its length. Then, the force on each electron being  $eE$ , we have to put

<sup>1)</sup> It may be observed that, as must be the case, the value (20) gives  $N$  for the number of electrons per unit volume and  $\frac{3}{2h}$  for the mean square of velocity

$$X = \frac{e E}{m}.$$

Also,

$$\frac{dA}{dx} = 0 \text{ and } \frac{dh}{dx} = 0,$$

so that (21) becomes

$$v = \frac{4\pi l A e}{3 h m} E.$$

Multiplying this by  $e$ , we find an expression for the electric current per unit area, and in order to find the coefficient of conductivity, we must finally divide by  $E$ . The result is

$$\sigma = \frac{4\pi l A e^2}{3 h m}, \quad . . . . . (23)$$

or, taking into account the relations (13) and (14) and denoting by  $u$  a velocity whose square is the mean square  $\frac{3}{2h}$  of the velocity of heat-motion,

$$\sigma = \sqrt{\frac{2}{3\pi}} \cdot \frac{l N e^2 u}{\alpha T} \quad . . . . . (24)$$

DRUDE gives the value

$$\sigma = \frac{1}{4} \frac{l N e^2 u}{\alpha T}.$$

§ 9. The determination of the coefficient of *conductivity for heat*, which we shall call  $k$  (expressing quantities of heat in mechanical units) is rather more difficult. This is due to the circumstance that, if initially  $X=0$ , the equation (21) implies the existence of an electric current in a bar whose parts are unequally heated. This current will produce a certain distribution of electric charges and will ultimately cease if the metal is surrounded on all sides by non-conductors. The final state will be reached when the difference of potential and the electric force arising from the charges have increased to such a degree that everywhere  $v=0$ .

Since it is this final state, with which one has to do in experiments on the conduction of heat, we shall calculate the flux of heat in the assumption that it has been established.

In the first place we have then by (21), putting  $v=0$ ,

$$2h A X - \frac{dA}{dx} = -2 \frac{A}{h} \frac{dh}{dx} \quad . . . . . (25)$$

and next, substituting this in (22) and again using the formula (14),

$$W = - \frac{8\pi l A \alpha}{9h^2} \frac{dT}{dx}.$$

Consequently, the coefficient of conductivity has the value

$$k = \frac{8\pi l A \alpha}{9h^2}, \quad . . . . . (26)$$

or

$$k = \frac{8}{9} \sqrt{\frac{2}{3\pi}} l N \alpha u, \quad . . . . . (27)$$

DRUDE's result for this case is

$$k = \frac{1}{3} l N \alpha u.$$

The ratio of the two conductivities is by my formulae

$$\frac{k}{\sigma} = \frac{8}{9} \left( \frac{\alpha}{e} \right)^2 T \quad . . . . . (28)$$

and by those of DRUDE

$$\frac{k}{\sigma} = \frac{4}{3} \left( \frac{\alpha}{e} \right)^2 T.$$

Here again, the difference between the two formulae consists merely in the numerical coefficients.

Just like DRUDE we may therefore conclude that the value of  $\frac{k}{\sigma}$  does not depend on the nature of the metal and that it varies proportionately to the absolute temperature, consequences that have been verified with a certain approximation in the case of many metals.

It need hardly be observed that these conclusions could only be arrived at because we have neglected the mutual encounters between electrons<sup>1)</sup>. In fact, these would tend to diminish the conductivity for heat, but not that for electricity, since they cannot have an influence in a phenomenon in which all electrons move in the same way. It is clear that, under these circumstances, a value of  $\frac{k}{\sigma}$  independent of the nature of the metal could hardly be expected.

Let us next consider the absolute values.

The value of  $\frac{\alpha T}{e}$  that can be deduced from those of  $k$  and  $\sigma$  and for which, using (28), I find

$$\frac{\alpha T}{e} = \sqrt{\frac{9}{8} \frac{k}{\sigma}} T, \quad . . . . . (29)$$

<sup>1)</sup> See THOMSON, l. c., p. 146.

may be compared, as has been observed by DRUDE and REINGANUM<sup>1)</sup>, to a value of the same expression that is obtained from other data. I shall suppose that the charge  $e$  of an electron is equal to that of an ion of hydrogen in an electrolytic solution and I shall represent by  $p$  the pressure that would be exerted, at the temperature  $T$ , by gaseous hydrogen, if a unit of volume contained one electrochemical equivalent. Then

$$\frac{\alpha T}{e} = 3p.$$

The proof of this formula is as follows. We may write for the number of atoms in unit volume of the gas considered  $\frac{1}{e}$ , for the number of molecules  $\frac{1}{2e}$ , and, since the mean kinetic energy of a molecule amounts to  $\alpha T$ , for the total kinetic energy  $\frac{\alpha T}{2e}$ . As is well known, the numerical value of the pressure per unit area is two thirds of this.

Using the C. G. S. system and electromagnetic units, we have for the electrochemical equivalent of hydrogen 0,000104 and, putting,  $T = 273^\circ + 18^\circ$ ,

$$3p = 38.$$

On the other hand, the measurements of JAEGER and DIESSELHORST have given for silver at  $18^\circ$  C.

$$\frac{k}{\sigma} = 6,85 \times 10^{10},$$

whence, by (29),

$$\frac{\alpha T}{e} = 47.$$

The agreement between the results of the two calculations, for which the data have been furnished by widely different phenomena, though not quite satisfactory, is close enough to make us feel confident that DRUDE's theory rests on a sound basis<sup>2)</sup>.

§ 10. We might now return to the formula (25) and, denoting by  $\varphi$  the electric potential, so that

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<sup>1)</sup> REINGANUM, Theoretische Bestimmung des Verhältnisses von Wärme- und Elektrizitätsleitung der Metalle aus der DRUDE'schen Elektronentheorie, Ann. Phys., 2 (1900), p. 398.

<sup>2)</sup> A better agreement is found if, instead of (28), we use DRUDE's formula.

$$X = -\frac{e}{m} \frac{d\varphi}{dx},$$

we might deduce from it expressions for the fall of potential in each point and for the difference of potential between the ends of the bar.

It is more interesting, however, to make a calculation of this kind for a more general case. Before doing so, we may observe that the equations (21) and (22) may be applied to a thin curved wire or bar and that we may as well suppose the normal section slowly to change from one point to another. The line passing through the centres of gravity of the normal sections may be called the axis of the conductor and we shall understand by  $x$  the distance from a fixed point, measured along this axis. We shall also assume that in all points of one and the same normal section the properties of the bar and the temperature are the same, but that, generally speaking, both depend on  $x$ , changing from one section to the next. By making different assumptions in this respect, we come to consider *circuits* of different kinds, composed of one or more metals and with any distribution of temperature we like.

For the sake of generality we shall introduce the notion of "molecular" forces of one kind or another exerted by the atoms of the metal on the electrons and producing for each electron a resulting force along the circuit in all points where the metal is not homogeneous. Actions of this nature have been imagined long ago by HELMHOLTZ for the purpose of explaining the phenomena of contact-electricity. We may judge of their effect in the simplest way by introducing the corresponding potential energy  $V$  of an electron relatively to the metallic atoms. This quantity, variable with  $x$  wherever the metal is not homogeneous, will be a constant in any homogeneous part of the circuit; we shall suppose this even to be so in case such a part is not uniformly heated. If, as before, we write  $\varphi$  for the electric potential, the force  $X$  divides into two parts

$$\left. \begin{aligned} X &= X_m + X_e, \\ X_m &= -\frac{1}{m} \frac{dV}{dx}, \quad X_e = -\frac{e}{m} \frac{d\varphi}{dx} \end{aligned} \right\} \dots \dots \dots (30)$$

We shall now consider an *open* circuit, calling the ends  $P$  and  $Q$ , and reckoning  $x$  from the former end towards the latter. Putting in (21)  $v = 0$  and attending to (30), we obtain for the stationary state

$$\frac{d\varphi}{dx} = -\frac{1}{e} \frac{dV}{dx} - \frac{m}{e} \frac{d}{dx} \left( \frac{1}{h} \right) - \frac{m}{2eh} \frac{d \log A}{dx}, \dots \dots (31)$$

whence by integration



$$\varphi_Q - \varphi_P = \frac{1}{e} (V_P - V_Q) + \frac{m}{e} \left( \frac{1}{h_P} - \frac{1}{h_Q} \right) - \frac{m}{2e} \int_P^Q \frac{1}{h} \frac{d \log A}{d x} d x, \quad . \quad . \quad (32)$$

a formula which may now be applied to some particular cases.

*a.* Let all parts of the circuit be kept at the same temperature. Then,  $h$  is a constant, and

$$\varphi_Q - \varphi_P = \frac{1}{e} (V_P - V_Q) + \frac{m}{2 e h} (\log A_P - \log A_Q) \quad . \quad (33)$$

The potential-difference will now have a positive or negative value, if the ends of the circuit are made of different metals. It appears in this way that the differences that have been observed in this case may be attributed either to an inequality of  $V_P$  and  $V_Q$ , i. e. to "molecular" forces acting at the places of junction (HELMHOLTZ), or to an inequality of  $A_P$  and  $A_Q$ , i. e. to a difference in the "concentrations" proper to the metals (DRUDE).

It need hardly be added that (33) becomes 0 whenever the ends are made of the same metal and that the law expressed in VOLTA's tension-series is implied by the equation.

*b.* Let the metal be the same everywhere. Then  $A$  is a function of  $h$  and (32) will always be 0, if the ends  $P$  and  $Q$  are kept at the same temperature, whatever be the distribution of temperature in the intermediate parts.

*c.* Let us next examine the potential-difference between the ends of an open thermo-electric circuit, a difference that may be regarded as the measure for the electromotive force  $F$  existing in it. Starting from  $P$  and proceeding towards  $Q$ , the state of things I shall consider is as follows: 1<sup>st</sup>. Between  $P$  and a section  $R'$ , the metal  $I$  maintained at a temperature varying from  $T_P$  to  $T'$  in  $R'$ . 2<sup>nd</sup> Between  $R'$  and  $S'$ , a gradual transition (§ 6) from the metal  $I$  to the metal  $II$ , at the uniform temperature  $T'$ . 3<sup>rd</sup> From  $S'$  to  $S''$ , the metal  $II$  with temperatures varying from  $T'$  to  $T''$ . 4<sup>th</sup> Between  $S''$  and  $R''$ , a gradual transition from the metal  $II$  to the metal  $I$ , the temperature being  $T''$  in every point of this part of the circuit. 5<sup>th</sup> Finally, between  $R''$  and  $Q$ , the metal  $I$  with a temperature changing from  $T''$  to  $T_Q = T_P$ . It being here implied that the ends of the circuit consist of the same metal and have the same temperature, the equation (32) reduces to the last term, and we find, after integration by parts,

$$F = \frac{m}{2e} \int_P^Q \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx \quad . \quad . \quad . \quad . \quad . \quad (34)$$

This integral may be divided into five parts, corresponding to the above parts of the circuit.

Distinguishing by appropriate indices the different values of  $h$  and  $A$  that have to be considered and keeping in mind that  $h$  is a constant both in the second and the fourth part, we have

$$\begin{aligned} \int_{R'}^{S'} \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx &= 0, \quad \int_{S''}^{R''} \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx = 0, \\ \int_P^{R'} \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx + \int_{R''}^Q \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx &= \int_{h''}^{h'} \log A_I \frac{d}{dh} \left( \frac{1}{h} \right) dh, \end{aligned}$$

$h'$  and  $h''$  being the values corresponding to  $T'$  and  $T''$ , the temperatures in  $R'$  and  $R''$ . Similarly

$$\int_{S'}^{S''} \log A \frac{d}{dx} \left( \frac{1}{h} \right) dx = \int_{h'}^{h''} \log A_{II} \frac{d}{dh} \left( \frac{1}{h} \right) dh.$$

If we combine these results, the formula (34) for the electromotive force becomes

$$F = \frac{m}{2e} \int_{h'}^{h''} \log \frac{A_I}{A_{II}} \frac{1}{h^2} dh,$$

or, if we use (13) and (14),

$$F = \frac{2\alpha}{3e} \int_{T'}^{T''} \log \frac{N_{II}}{N_I} dT. \quad . \quad . \quad . \quad . \quad . \quad (35)$$

**Geodesy.** — “*The connection between the primary triangulation of South-Sumatra and that of the West Coast of Sumatra.*” By Mr. S. BLOK. (Communicated by Prof. J. A. C. OUDEMANS).

I. *Short description of the triangulations of South-Sumatra and the West Coast of Sumatra*<sup>1)</sup>.

Towards the end of 1896 the measurements for the primary triangulation, which will serve as a basis for the topographical sur-

<sup>1)</sup> For a more detailed description I refer to the papers of Dr. J. J. A. MULLER, occurring in the proceedings of the International Geodetic Association of 1892, 1896 and 1903.