KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday January 28, 1905.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 Januari 1905, Dl. XIII).

CONTENTS.

The following papers were read:

Mathematics. — "*The formulae of* Guldin *in polydimensional space.*" By Prof. P. H. Schoute.

(Communicated in the meeting of December 24, 1904).

We suppose in space S_n with n dimensions an axial space $S_p^{(a)}$ and in a space S_{p+1} through this $S_p^{(a)}$ a limited part with $p+1$ dimensions rotating round $S_p^{(a)}$. Then an arbitrary point P of this limited space, which may be called a polytope independent of the shape of its limitation and may be represented by the symbol $(Po)_{p+1}$, describes a spherical space of $n-p$ dimensions lying in the space S_{n-p} through P perpendicular to $S_p^{(a)}$ having the projection Q of P

on $S_p^{(a)}$ as centre, PQ as radius; so it can be represented by the symbol $Sp_{n-p}(Q, PQ)$.

The question with which we shall occupy ourselves is as follows: "How do we determine volume and surface of the figure of revolution generated by $(Po)_{p+1}$ rotating round $S_p^{(a)}$ if we assume that $(Po)_{p+1}$ and $S_p^{(a)}$ though lying in the same space S_{p+1} have no points in common?"

This theorem is solved with the aid of a simple extension of the well known formulae of GULDIN, which serve in our space to determine the volume and the surface of a *figure of revolution*. To prove these generalized formulae we have but to know that the surface of the above-mentioned spherical space $Sp_{n-p}(Q, PQ)$ is found by multiplying $\overline{PQ}^{n-p-1}$ by a coefficient s_{n-p} only dependent on $n-p$; for its application however it is desirable to know not only this coefficient of surface s_{n-p} but also the coefficient of volume v_{n-p} by which $\overline{PQ}^{n-p}$ must be multiplied to arrive at the volume of the same spherical space. To this end we mention beforehand — as is learned by direct integration — that between these coefficients the recurrent relations

$$v_n = \frac{2\pi}{n} v_{n-2} \quad , \quad s_n = \frac{2\pi}{n-2} s_{n-2} \quad . \quad . \quad . \quad . \quad (1)$$

exist, whilst the well known relation between volume and surface leads in a simpler way still to the equation

$$v_n = \frac{1}{n} s_n \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this way we find as far as and inclusive of $n = 12$ out of the well known values of v_2, v_3 and s_2, s_3

n	2	3	4	5	6	7	8	9	10	11	12
v_n	π	$\frac{4}{3}\pi$	$\frac{1}{2}\pi^3$	$\frac{8}{15}\pi^2$	$\frac{1}{6}\pi^3$	$\frac{16}{105}\pi^3$	$\frac{1}{24}\pi^4$	$\frac{32}{945}\pi^4$	$\frac{1}{120}\pi^5$	$\frac{64}{10395}\pi^5$	$\frac{1}{720}\pi^6$
s_n	2π	4π	$2\pi^2$	$\frac{8}{3}\pi^2$	π^3	$\frac{16}{15}\pi^3$	$\frac{1}{3}\pi^4$	$\frac{32}{105}\pi^4$	$\frac{1}{12}\pi^5$	$\frac{64}{945}\pi^5$	$\frac{1}{60}\pi^6$

1. Determination of volume. If a indicates the length of the radius PQ and the differential dv the $p+1$-dimensional volume-element, lying immediately round P, of the rotating polytope $(Po)_{p+1}$, then the demanded volume is

$$V = s_{n-p} \int x^{n-p-1}\, dv,$$

if the integral is extended to all the elements of volume of $(Po)_{p+1}$. If now V_{p+1} is the volume of $(Po)_{p+1}$, we can imagine a quantity $\bar{x}$, satisfying the equation

$$\int x^{n-p-1}\, dv = \bar{x}^{n-p-1} \int dv = \bar{x}^{n-p-1}\, V_{p+1}$$

and we can insert this quantity in the above formula. By this it passes into

$$V = V_{p+1} \cdot s_{n-p}\, \bar{x}^{n-p-1}.$$

If we call $\bar{x}$ the "radius of inertia of order $n-p-1$" of the volume V_{p+1} of the rotating figure $(Po)_{p+1}$ with relation to the axial space $S_p^{(a)}$ lying in its space S_{p+1}, we find this theorem:

"We find the volume of the figure of revolution generated by the polytope $(Po)_{p+1}$ rotating round an axial space $S_p^{(a)}$ not cutting this polytope of its space S_{p+1}, if we multiply the volume V_{p+1} of $(Po)_{p+1}$ by the surface of a spherical space Sp_{n-p}, having the radius of inertia of order $n-p-1$ of V_{p+1} with relation to $S_p^{(a)}$ as radius."

2. Determination of surface. If in the above we substitute the p-dimensional element of surface for the $p+1$-dimensional element of volume and in accordance with this for the volume V_{p+1} and its radius of inertia the surface Su_{p+1} and its radius of inertia, we arrive in similar way at the theorem:

We find the surface of the figure of revolution generated as above if we multiply the surface Su_{p+1} of $(Po)_{p+1}$ by the surface of a spherical space Sp_{n-p}, having for radius the radius of inertia of order $n-p-1$ of Su_{p+1} with relation to $S_p^{(a)}$.

3. The segment of revolution. The opinions will differ greatly about the use of the n-dimensional extension of the GULDIN formulae proved above. Those regarding only their generality and their short enunciation may rate them too high, though reasonably they cannot go so far as to believe that these formulae allow the volume and the surface of a figure of revolution to be found when the common principles of the calculus leave us in the lurch, as the quadratures can be indicated but not effected in finite form.

Others, whose attention is drawn to the fact that these formulae displace the difficulties of the quadratures but apparently — in this case displace them from definition of volume and of surface to the definition of radii of inertia — will on the other hand perhaps fall into another extreme and will deny any practical use to the formulae in question. Here of course the truth lies in the mean. Though it remains true that the GULDIN formulae help us but apparently out of the difficulty in the case where the direct integration falls short, yet by the use of those formulae many an integration is avoided because the radii of inertia appearing in those formulae of volume and surface of the figure of revolution are known from another source, which latter circumstance appears in the first place when $p = n - 2$, thus each point P of the rotating figure describes the circumference of a circle and the radii of inertia relate therefore to the centre of gravity of volume and surface of that figure, whilst for $p = n - 3$ the knowledge of the common radius of inertia of mechanics gives rise to simplification.

As simplest example of the case $p = n - 2$ we think that a segment $Sp_{n-1}(r, \varrho)$ of a spherical space Sp_{n-1} with r and ϱ as radii of spherical and base boundary generates a segment of revolution $Sp(r, \varrho, \alpha)_n$ by rotation round a diametral space $S^{(a)}_{n-2}$, situated in its space S_{n-1}, having no point in common with it and forming an angle α with the space S_{n-2} of the base boundary. For this we find the following theorems:

"We find the volume of the segment of revolution $Sp(r, \varrho, \alpha)_n$ by multiplying the volume of a spherical space Sp_n with ϱ for radius by *cos* α."

"We find the surface of the segment of revolution $Sp(r, \varrho, \alpha)_n$ which is described by the spherical boundary of $Sp_{n-1}(r, \varrho)$ when rotating, by multiplying the circumference of a circle with r for radius by the volume of the projection of the base boundary of $Sp_{n-1}(r, \varrho)$ on the axial space $S^{(a)}_{n-2}$."

These theorems are simple polydimensional extensions of well known theorems of stereometry. They can be found by direct integration where the case $\alpha = 0$ is considerably simpler than that of an arbitrary angle α. And now the formulae of GULDIN teach us exactly to avoid the integration in the general case, showing us immediately that the theorems are true for the case of an arbitrary angle α, as soon as they are proved for $\alpha = 0$. If namely x_v and x_s are the distances from the centres of gravity of volume V_{n-1} and surface

Su_{n-1} of $Sp_{n-1}(r, \varrho)$ to $S_{n-2}^{(a)}$, where Su_{n-1} now again indicates exclusively the spherical boundary, then the formulae of GULDIN furnish us with

$$\left. \begin{array}{l} V_\alpha = 2\pi x_v \cos\alpha \, . \, V_{n-1} \\ V_0 = 2\pi x_v \quad\quad . \, V_{n-1} \end{array} \right\} , \quad \left. \begin{array}{l} Su_\alpha = 2\pi x_s \cos\alpha \, . \, Su_{n-1} \\ Su_0 = 2\pi x_s \quad\quad . \, Su_{n-1} \end{array} \right\}$$

and from this ensues immediately

$$V_\alpha = V_0 \cos\alpha \quad , \quad Su_\alpha = Su_0 \cos\alpha$$

and therefore what was assumed above, so that only for $\alpha = 0$ the proofs have yet to be given. We commence with the volume. If x is the distance from $S_{n-2}^{(a)}$ to a parallel space $S_{n-2}^{(x)}$ cutting $Sp_{n-1}(r, \varrho)$ in a spherical space $Sp_{n-2}^{(x)}$ with $y = \sqrt{r^2 - x^2}$ for radius, then the demanded volume is

$$V = 2\pi\, v_{n-2} \int_{x=\sqrt{r^2-\rho^2}}^{x=r} y^{n-2}\, x dx$$

and this passes, as $x^2 + y^2 = r^2$ and $xdx + ydy = 0$, into

$$V = 2\pi\, v_{n-2} \int_0^\rho y^{n-1}\, dy = \frac{2\pi}{n} v_{n-2}\, \varrho^n = v_n\, \varrho^n,$$

with which the special case of the theorem for the volume has been proved.

In the special case of the theorem for the surface we regard the superficial element generated by the rotation of the surface $Su_{n-1}(r, \varrho)$ situated between the parallel spaces $S_{n-2}^{(x)}$ and $S_{n-2}^{(x+dx)}$. If ds is the apothema of this frustum the demanded surface is

$$Su = 2\pi\, s_{n-2} \int_{x=\sqrt{r^2-\rho^2}}^{x=r} y^{n-3}\, x ds.$$

With the help of the relations $yds = rdx$ and $xdx + ydy = 0$ this passes into

$$Su = 2\pi r\, s_{n-2} \int_0^\rho y^{n-3} dy = \frac{2\pi}{n-2} r\, s_{n-2}\, \varrho^{n-2} = 2\pi r \,.\, v_{n-2}\, \varrho^{n-2},$$

i. e. the desired result.

Of course we can represent to ourselves the more general segment of revolution $Sp(r, \varrho, a)_{n,k}$ of order k generated by the rotation of a spherical segment $Sp_{n-k}(r,\varrho)$ round a diametral space $S^{(a)}_{n-k-1}$ of its space S_{n-k}; of the various possible cases

$$k = 1, 2, \ldots\ldots\ldots, n-2$$

the first is the one treated above extensively. As any point generates at the rotation the surface of a spherical space Sp_{k+1}, we find — if along the indicated way by means of the formulae of GULDIN the general case of an arbitrary angle a is reduced to the special case $a = 0$ — for volume $V_{n,k}$ and the surface $Su_{n,k}$ of $Sp\,(r, \varrho, a)_{n,k}$ the formulae

$$\left.\begin{aligned} V_{n,k} &= v_{n-k-1}\, s_{k+1} \cos^k a \int_{x=\sqrt{r^2-\rho^2}}^{x=r} y^{n-k-1}\, x^k\, dx \\ Su_{n,k} &= r s_{n-k-1}\, s_{k+1} \cos^k a \int_{x=\sqrt{r^2-\rho^2}}^{x=r} y^{n-k-3}\, x^k\, dx \end{aligned}\right\}$$

and from this ensues the general relation

$$Su_{n,k} = 2\pi\, r \cos^2 a\ V_{n-2,k},$$

by which all cases of determination of surface except $Su_{n,n-2}$ and $Su_{n,n-3}$ are deduced to simpler cases of the determination of volume.

When determining the volume the integral gives a rational result, an irrational one or a transcendental one according to k being odd, n odd and k even, or n even and k even. And this is evidently likewise the case for the determination of surface.

4. The torusgroup. By rotation of a spherical space $Sp_{n-k}(r)$ around a space $S^{(a)}_{n-k-1}$ of its space S_{n-k} at a distance $a > r$ from the centre a ring is generated in S_n, the ring or "torus" $T(r,a)_{n,k}$. For volume $V(r,a)_{n,k}$ and surface $Su(r,a)_{n,k}$ of this figure of revolution of order k we find

$$\left.\begin{aligned} V(r,a)_{n,k} &= s_{k+1}\, v_{n-k-1} \int_{-a}^{a} \sqrt{r^2-x^2}^{\,n-k-1} (a+x)^k dx \\ Su(r,a)_{n,k} &= r\, s_{k+1}\, s_{n-k-1} \int_{-a}^{a} \sqrt{r^2-x^2}^{\,n-k-3} (a+x)^k dx \end{aligned}\right\} \quad . \;\; . \;\; (3),$$

from which ensues again the formula of reduction

$$Su_{n,k} = 2\,\pi\, r\, V_{n-2,k} \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (4)$$

For the case $k=1$ and $k=2$ the results are calculated more easily by means of the formulae of GULDIN, if one makes use of the centre of gravity and of the oscillation centre of the rotating spherical space.

Case $k=1$. The centre of gravity of volume and surface of the spherical space $Sp_{n-1}(r)$ lying in the centre, we find

$$V=2\pi a \ v_{n-1}r^{n-1} \quad , \quad Su=2\pi a . s_{n-1}r^{n-2}.$$

Case $k=2$. The radii of inertia of volume and surface of a spherical space $Sp_{n-2}(r)$ with respect to the centre are $r\sqrt{\frac{n-2}{n}}$ and r, those with respect to a diametral space S_{n-3} are thus $r\sqrt{\frac{1}{n}}$ and $r\sqrt{\frac{1}{n-2}}$. So we find

$$V=4\pi\left(a^2+\frac{1}{n}r^2\right)v_{n-2}r^{n-2} \quad , \quad Su=4\pi\left(a^2+\frac{1}{n-2}r^2\right).s_{n-2}r^{n-3}.$$

If instead of a whole spherical space $Sp_{n-k}(r)$ we allow only half of it to rotate around a space $S^{(a)}_{n-k-1}$ in its space S_{n-k} parallel to its base at a distance a, then the limits $(-r,r)$ of the two integrals (1) change into $(0,r)$ or $(-r,0)$ according to the half spherical space $Sp_{n-k}(r)$ turning its base or its spherical boundary to the axial space $S^{(a)}_{n-k-1}$. We shall occupy ourselves another moment with the former of these cases, namely for $k=1$ and $k=2$.

Case $(0,r)$, $k=1$. We find immediately

$$V=\pi\left(a+\frac{2}{n}\frac{v_{n}}{v_{n-1}}\frac{2}{}r\right).v_{n-1}r^{n-1} \quad , \quad Su=\pi\left(a+\frac{2}{n-2}\frac{s_{n-2}}{s_{n-1}}r\right).s_{n-1}r^{n-2}.$$

Case $(0,r)$, $k=2$. We determine the moments of inertia of volume and surface first with respect to the base $S^{(b)}_{n-3}$ and then successively with respect to the parallel space $S^{(z)}_{n-3}$ through the centre of gravity and with respect to the axial space $S^{(a)}_{n-3}$. Thus we finally find the formulae

$$V=2\pi\left\{\frac{r^2}{n}-\left(\frac{2}{n}\frac{v_{n-2}}{v_{n-1}}r\right)^2+\left(\frac{2}{n}\frac{v_{n-2}}{v_{n-1}}r+a\right)^2\right\}.v_{n-2}r^{n-2},$$

$$Su=2\pi\left\{\frac{r^2}{n-2}-\left(\frac{2}{n-2}\frac{s_{n-2}}{s_{n-1}}r\right)^2+\left(\frac{2}{n-2}\frac{s_{n-2}}{s_{n-1}}r+a\right)^2\right\}.s_{n-2}r^{n-3},$$

or

$$V=2\pi\left(a^2+\frac{4}{n}\frac{v_{n-2}}{v_{n-1}}ar+\frac{r^2}{n}\right).v_{n-2}r^{n-2},$$

$$Su=2\pi\left(a^2+\frac{4}{n-2}\frac{s_{n-2}}{s_{n-1}}ar+\frac{r^2}{n-2}\right).s_{n-2}r^{n-3},$$

which pass for $a=0$ appropriately into volume and surface of the spherical space $Sp_n(r)$.