Citation:

Kapteyn, W., On a series of Bessel functions, in:
KNAW, Proceedings, 7, 1904-1905, Amsterdam, 1905, pp. 494-500

Mathematics. — "*On a series of Bessel functions.*" By Prof. W. KAPTEYN.

(Communicated in the meeting of December 24, 1904).

In the following we shall try to determine the sum of the series

$$I_1(a)\, I_1(x) + 3\, I_3(a)\, I_3(x) + 5\, I_5(a)\, I_5(x) + \ldots = \sum_{1.3}^{\infty} n\, I_n(a)\, I_n(x).$$

To this end we begin to determine the sum of the simpler series

$$S = \sum_{1\,3}^{\infty} I_n(x) \cos n\varphi .$$

If we introduce, n being an odd number, for the Bessel function the form

$$I_n(x) = - \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} t^{n-1},$$

then

$$S = - \mathcal{E}_0 \frac{e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}}{t} (t \cos \varphi + t^3 \cos 3\varphi + \ldots),$$

and

$$t \cos \varphi + t^3 \cos 3\varphi + \ldots = \frac{t\,(1 - t^2) \cos \varphi}{1 - 2\, t^2 \cos 2\varphi + t^4}\ (mod\, t < 1),$$

hence

$$S = - \mathcal{E}_0 \frac{e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2) \cos \varphi}{1 - 2\, t^2 \cos 2\varphi + t^4}.$$

If we put

$$R = \frac{\cos \varphi}{1 - 2\, t^2 \cos 2\varphi + t^4},$$

then

$$S = - \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2)\, R,$$

or

$$\sum_{1\,3}^{\infty} I_n \cos n\varphi = - \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2)\, R.$$

Differentiating this equation, we get

$$\sum_{1.3}^{\infty} n\, I_n(x) \sin n\varphi = \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2) \frac{dR}{d\varphi}.$$

If now we multiply this equation by $\frac{1}{\pi} \sin (\alpha \sin \varphi)\, d\varphi$ and if we integrate between the limits 0 and π we find

$$\sum_{1\,3}^{\infty} n\, I_n(x)\, I_n(\alpha) = \frac{1}{\pi} \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2) \int_0^{\pi} \frac{dR}{d\varphi} \sin (\alpha \sin \varphi)\, d\varphi$$

$$= -\frac{\alpha}{\pi} \mathcal{E}_0\, e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} (1 - t^2) \int_0^{\pi} R \cos (\alpha \sin \varphi) \cos \varphi\, d\varphi.$$

Putting for the further reduction

$$u = \int_0^{\pi} R \cos (\alpha \sin \varphi) \cos \varphi\, d\varphi$$

$$= \int_0^{\pi} \frac{\cos^2 \varphi \cos (\alpha \sin \varphi)}{1 - 2\, t^2 \cos 2\varphi + t^4}\, d\varphi$$

we arrive at

$$\frac{du}{d\alpha} = -\int_0^{\pi} \frac{\cos^2 \varphi \sin (\alpha \sin \varphi)}{1 - 2\, t^2 \cos 2\varphi + t^4} \sin \varphi\, d\varphi,$$

$$\frac{d^2 u}{d\alpha^2} = -\int_0^{\pi} \frac{\cos^2 \varphi \cos (\alpha \sin \varphi)}{1 - 2\, t^2 \cos 2\varphi + t^4} \sin^2 \varphi\, d\varphi$$

and because

$$\sin^2 \varphi = \frac{1 - 2\, t^2 \cos 2\varphi + t^4}{4\, t^2} - \frac{(1 - t^2)^2}{4\, t^2},$$

we find

$$\frac{d^2 u}{d\alpha^2} = m^2\, u - \frac{1}{4\, t^2} \int_0^{\pi} \cos^2 \varphi \cos (\alpha \sin \varphi)\, d\varphi,$$

where $m = \frac{1 - t^2}{2t}$.

If we replace $\cos^2 \varphi$ by $\frac{1 + \cos 2\varphi}{2}$, we can easily reduce this differential equation to

$$\frac{d^2 u}{d\alpha^2} - m^2\, u = -\frac{\pi}{8t^2} (I_0(\alpha) + I_2(\alpha))$$

$$= -\frac{\pi}{4t^2} \frac{I_1(\alpha)}{\alpha}.$$

Let us now determine the integral of this equation satisfying the conditions that for $\alpha = 0$

$$u = \int_0^\pi \frac{\cos^2 \varphi \, d\varphi}{1 - 2t^2 \cos^2 \varphi + t^4} = \frac{\pi}{2(1-t^2)},$$

and

$$\frac{du}{d\alpha} = 0.$$

We then find

$$u = \frac{\pi}{4(1-t^2)}\left[e^{m\alpha} + e^{-m\alpha}\right] - \frac{\pi}{8t^2 m}\int_0^\alpha \frac{I_1(\beta)}{\beta} d\beta \left[e^{m(\alpha-\beta)} - e^{-m(\alpha-\beta)}\right]$$

and by this

$$\sum_{1\,3}^{\infty} n\, I_n(\alpha)\, I_n(x) = -\frac{\alpha}{4}\, \mathcal{E}_0\left[e^{\frac{x-\alpha}{2}\left(t-\frac{1}{t}\right)} + e^{\frac{x+\alpha}{2}\left(t-\frac{1}{t}\right)}\right] +$$

$$+ \frac{\alpha}{4}\int_0^\tau \frac{I_1(\beta)}{\beta} d\beta\, \mathcal{E}_0 \frac{1}{t}\left[e^{\frac{x-\alpha+\beta}{2}\left(t-\frac{1}{t}\right)} - e^{\frac{x+\alpha-\beta}{2}\left(t-\frac{1}{t}\right)}\right].$$

Remembering now that

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = I_0(z) + t\, I_1(z) + t^2 I_2(z) + \dots$$
$$- \frac{1}{t} I_1(z) + \frac{1}{t^2} I_2(z) - \dots$$

we see that the residues are easily determined. We have

$$\sum_{1.3}^{\infty} n\, I_n(\alpha)\, I_n(x) = \frac{\alpha}{4}\left[I_1(x-\alpha) + I_1(x+\alpha)\right] +$$

$$+ \frac{\alpha}{4}\int_0^\alpha \frac{I_1(\beta)}{\beta} d\beta \left[I_0(x-\alpha+\beta) - I_0(x+\alpha-\beta)\right]. \quad (1)$$

From this result another important relation may be deduced. To show this, we shall again develop

$$I_1(x-\alpha) + I_1(x+\alpha)$$

into a series.

From

$$J_1(x-\alpha) = \frac{1}{\pi}\int_0^\pi \sin\varphi \sin(x \sin\varphi - \alpha \sin\varphi)\, d\varphi$$

and

$$I_1(x+a) = \frac{1}{\pi}\int_0^\pi \sin\varphi \sin(x\sin\varphi + a\sin\varphi)\, d\varphi$$

follows

$$I_1(x-a) + I_1(x+a) = \frac{2}{\pi}\int_0^\pi \sin\varphi \sin(x\sin\varphi)\cos(a\sin\varphi)\, d\varphi.$$

If we write

$$\sin(x\sin\varphi) = 2\, I_1 \sin\varphi + 2\, I_3 \sin 3\varphi + \ldots$$

we obtain

$$\begin{aligned} I_1(x-a) + I_1(x+a) = & \frac{4}{\pi} I_1(x)\int_0^\pi \sin^2\varphi \cos(a\sin\varphi)\, d\varphi \\ & + \frac{4}{\pi} I_3(x)\int_0^\pi \sin\varphi \sin 3\varphi \cos(a\sin\varphi)\, d\varphi \\ & + \frac{4}{\pi} I_5(x)\int_0^\pi \sin\varphi \sin 5\varphi \cos(a\sin\varphi)\, d\varphi \\ & + \ldots . \end{aligned}$$

or as

$$\begin{aligned} 2\int_0^\pi \sin\varphi \sin(2n+1)\varphi \cos(a\sin\varphi)\, d\varphi = & \\ & = \int_0^\pi [\cos 2n\varphi - \cos(2n+2)\varphi]\cos(a\sin\varphi)\, d\varphi \\ & = \pi\,[I_{2n}(a) - I_{2n+2}(a)] \\ & = 2\pi \frac{d\, I_{2n+1}(a)}{da} \end{aligned}$$

we get

$$I_1(x-a) + I_1(x+a) = 4\left[I_1(x)\frac{dI_1}{da} + I_3(x)\frac{dI_3}{da} + I_5(x)\frac{dI_5}{da}\ldots\right].$$

Substituting here

$$a\frac{dI_n}{da} = n\, I_n(a) - a\, I_{n+1}(a)$$

we arrive finally at

$$I_1(x-a) + I_1(x+a) = \frac{4}{\pi}\sum_{1,3}^{\infty} n\, I_n(a)\, I_n(x) - 4\sum_{1,3}^{\infty} I_{n+1}(a)\, I_n(x).$$

With this equation the result (1) may be written

$$\sum_{1\,3}^{\infty} I_{n+1}(\alpha)\, I_n(x) = \frac{1}{4}\int_0^{x} \frac{I_1(\beta)}{\beta}\, d\beta\, [I_0(x-\alpha+\beta) - I_0(x+\alpha-\beta)] \quad (2)$$

If here we develop

$$I_0(x-\alpha+\beta) = I_0(x)\, I_0(\alpha-\beta) + 2 I_1(x)\, I_1(\alpha-\beta) + 2 I_2(x)\, I_2(\alpha-\beta) + \dots$$

$$I_0(x+\alpha-\beta) = I_0(x)\, I_0(\alpha-\beta) - 2 I_1(x)\, I_1(\alpha-\beta) + 2 I_2(x)\, I_2(\alpha-\beta) - \dots$$

we find

$$\sum_{1.3}^{\infty} I_{n+1}(\alpha)\, I_n(x) = \sum_{1\,3}^{\infty} I_n(x) \int_0^{\alpha} I_n(\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta$$

and consequently by comparing the coefficients of $I_n(x)$

$$I_{n+1}(\alpha) = \int_0^{\alpha} I_n(\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By means of this formula we can give equation (1) another form. For, according to (3),

$$I_1(x-\alpha) = \int_0^{x-\alpha} I_0(x-\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta$$

$$= -\int_0^{\alpha-x} I_0(x-\alpha+\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta$$

$$I_1(x+\alpha) = \int_0^{x+\alpha} I_0(x+\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta$$

hence the second member of (1) takes the form

$$\frac{\alpha}{4}\left[-\int_0^{\alpha-x} I_0(x-\alpha+\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta + \int_0^{x+\alpha} I_0(x+\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta\right]$$

$$+\frac{\alpha}{4}\left[\int_0^{\alpha} I_0(x-\alpha+\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta - \int_0^{\alpha} I_0(x+\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta\right]$$

or

$$\frac{\alpha}{4}\left[\int_{\alpha-x}^{x} I_0(x-\alpha+\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta + \int_{\alpha}^{\alpha+x} I_0(x+\alpha-\beta)\, \frac{I_1(\beta)}{\beta}\, d\beta\right].$$

If we now put in the first integral $\beta = \alpha - \gamma$ and in the second one $\beta = \alpha + \gamma$ this becomes

$$\frac{\alpha}{4}\left[\int_0^x I_0(x-\gamma)\frac{I_1(\alpha-\gamma)}{\alpha-\gamma}d\gamma + \int_0^x I_0(x-\gamma)\frac{I_1(\alpha+\gamma)}{\alpha+\gamma}d\gamma\right],$$

with which equation (1) assumes the final form

$$\sum_{1\,3}^{\infty} n\, I_n(\alpha)\, I_n(x) = \frac{\alpha}{4}\int_0^x I_0(x-\gamma)\left[\frac{I_1(\alpha-\gamma)}{\alpha-\gamma} + \frac{I_1(\alpha+\gamma)}{\alpha+\gamma}\right]d\gamma. \quad . . (4)$$

A closer investigation of formula (3) teaches us, that it holds good for even values of n too, also that many analogous relations exist. So we find inter alia, k being any integer,

$$\int_0^\alpha \frac{I_n(\alpha-\beta)}{\alpha-\beta} I_1(\beta)\, d\beta = \frac{I_{n+1}(\alpha)}{n},$$

$$\int_0^\alpha I_n(\alpha-\beta)\frac{I_k(\beta)}{\beta}\, d\beta = \frac{I_{n+k}(\alpha)}{k},$$

$$\int_0^\alpha \frac{I_n(\alpha-\beta)}{\alpha-\beta} I_k(\beta)\, d\beta = \frac{I_{n+k}(\alpha)}{n},$$

$$\int_0^\alpha \frac{I_n(\alpha-\beta)}{(\alpha-\beta)^2} I_1(\beta)\, d\beta = \frac{1}{2n}\left[\frac{I_n(\alpha)}{n-1} + \frac{I_{n+2}(\alpha)}{n+1}\right],$$

$$\int_0^\alpha I_0(\alpha-\beta)\, I_0(\beta)\, d\beta = \sin \alpha.$$

We shall not dwell upon this at present; we only remark, that when a very great positive value is assigned in (1) to x, so that

$$I_n(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{2n+1}{4}\pi\right),$$

we find

$$I_1(x-\alpha) + I_1(x-\alpha) = 2\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}\right)\cos\alpha,$$

$$I_0(x-\alpha+\beta) - I_0(x+\alpha-\beta) = 2\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right)\sin(\alpha-\beta).$$

This changes (1) into

$$\sum_{1,3}^{\infty} n\, I_n(\alpha)\sin\frac{n\pi}{2} = \frac{\alpha}{2}\cos\alpha + \frac{\alpha}{2}\int_0^\alpha \frac{I_1(\beta)}{\beta}\sin(\alpha-\beta)\, d\beta$$

or, noticing that

$$\sum_{1,3}^{\infty} n \, I_n(\alpha) \sin \frac{n\pi}{2} = \frac{\alpha}{2} I_0(\alpha),$$

we have

$$I_0(\alpha) = \cos \alpha + \int_0^{\alpha} \frac{I_1(\beta)}{\beta} \sin(\alpha - \beta) \, d\beta$$

If we differentiate this equation, we find

$$I_1(\alpha) = \sin \alpha - \int_0^{\alpha} \frac{I_1(\beta)}{\beta} \cos(\alpha - \beta) \, d\beta$$

from which we conclude that

$$\int_0^{\alpha} \frac{I_1(\beta)}{\beta} \sin \beta \, d\beta = 1 - \cos \alpha \, I_0(\alpha) - \sin \alpha \, I_1(\alpha),$$

$$\int_0^{\alpha} \frac{I_1(\beta)}{\beta} \cos \beta \, d\beta = \sin \alpha \, I_0(\alpha) - \cos \alpha \, I_1(\alpha).$$

Geology. — *"Contributions to the knowledge of the sedimentary boulders in the Netherlands. 1. The Hondsrug in the province of Groningen. 2. Upper Silurian boulders. — First communication: Boulders of the age of the Eastern Baltic zone G."* By Dr. H. G. JONKER. (Communicated by Prof. K. MARTIN).

This communication introduces the description of the Upper Silurian boulders of Groningen and its surroundings, in which my contribution that treats of the Cambrian and Lower Silurian erratics and appeared in 1904, is continued (36). The circumstance that in the summer of last year I had an opportunity of getting more intimately acquainted with the Scandinavian-Baltic strata by investigations of my own has aided me considerably in the study of these younger rocks. Owing to nearly a month's stay in Gothland I managed to collect a great number of different species of rocks together with fossils characteristic to them in order to compare them with erratics that are found here. Much I owe to the kindness and assistance of Drs. O. W. WENNERSTEN, who accompanied me on some excursions and whom I had very often reason to admire for his extensive knowledge of his native country, the classical ground for the study of the Upper Silurian formation. But I have as yet not been able