Mathematics. — "*The equations by which the locus of the principal axes of a pencil of quadratic surfaces is determined*" by Mr. CARDINAAL.

1. The communication following here can be regarded as a continuation of the preceding one included in the Proceedings of Nov. 26 1904. It contains the analytical treatment of the problem, of which a geometrical treatment is given there. It ought to have been conducive to the finding of a surface of order nine; this has not been effected on account of the calculations becoming too extensive; however, the form of the final equation has been found.

2. In the first place the equation must be found of the cone of axes of the concentric pencil of quadratic cones, at the same time director cone of the locus of the axes of the pencil of surfaces. To this end we regard the intersection of the two cones, determining the pencil of cones, with the plane at infinity P_∞, and besides the isotropic circle situated in this plane; then we have the three equations in rectangular cartesian coordinates:

$$A \equiv a_{11}\,x^2 + a_{22}\,y^2 + a_{33}\,z^2 + 2a_{12}\,xy + 2a_{13}\,xz + 2a_{23}\,yz = 0,$$
$$B \equiv b_{11}\,x^2 + b_{22}\,y^2 + b_{33}\,z^2 + 2b_{12}\,xy + 2b_{13}\,xz + 2b_{23}\,yz = 0,$$
$$C \equiv x^2 + y^2 + z^2 = 0.$$

Out of these equations we find that of the cone of axes in the same way as we determine the Jacobian curve of a net of conics:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0,$$

where A_1, A_2, A_3, etc. are the derivatives of A with respect to x, y, z.

So the equation of the cone becomes

$$\begin{vmatrix} x & a_{11}\,x + a_{12}\,y + a_{13}\,z & b_{11}\,x + b_{12}\,y + b_{13}\,z \\ y & a_{12}\,x + a_{22}\,y + a_{23}\,z & b_{12}\,x + b_{22}\,y + b_{23}\,z \\ z & a_{13}\,x + a_{23}\,y + a_{33}\,z & b_{13}\,x + b_{23}\,y + b_{33}\,z \end{vmatrix} = 0.$$

Without harming the generality we can always assume that the principal axes of one of the cones coincide with the axes of coordinates; from this ensues that we may put $b_{12} = b_{13} = b_{23} = 0$, by which the equation of the cone is simplified.

3. After having found the equation of this cone we can pass to the formation of the set of equations, by means of which is found the equation of the locus of the axes,

The equation of the pencil of quadratic surfaces now becomes

$$A + \lambda B = 0, \quad \ldots \ldots \ldots \quad (1)$$

where however A and B have a wider meaning than before, A being

$$a_{11} x^2 + a_{22} y^2 + a_{33} z^2 + 2a_{12} xy + 2a_{13} xz + 2a_{23} yz + \\ + 2a_{14} x + 2a_{24} y + 2a_{34} z + a_{44},$$

and B being the same expression with the coefficients b.

Let us now put the coordinates of the centre of the surface (1) p, q, r and let us regard this centre as origin O' of a new system of coordinates with axes parallel to the original ones. We then arrive for surface (1) at an equation in x', y', z', in which the terms of order one are missing and those of order two possess the same coefficients. The principal axes of this surface are given by the three equations:

$$(a_{11} x' + a_{12} y' + a_{13} z') + \lambda (b_{11} x' + b_{12} y' + b_{13} z') + kx' = 0,$$
$$(a_{12} x' + a_{22} y' + a_{23} z') + \lambda (b_{12} x' + b_{22} y' + b_{23} z') + ky' = 0,$$
$$(a_{13} x' + a_{23} y' + a_{33} z') + \lambda (b_{13} x' + b_{23} y' + b_{33} z') + kz' = 0.$$

As could be foreseen the elimination of λ and k furnishes the same equation as was already found for the cone of axes.

If we wish to form the equation with respect to the original system of axes, we must put $x' = x - p$, $y' = y - q$, $z' = z - r$ and make use of the equations of condition for p, q, r:

$$\left.\begin{array}{l}(a_{11} + \lambda b_{11}) p + (a_{12} + \lambda b_{12}) q + (a_{13} + \lambda b_{13}) r + a_{14} + \lambda b_{14} = 0, \\ (a_{12} + \lambda b_{12}) p + (a_{22} + \lambda b_{22}) q + (a_{23} + \lambda b_{23}) r + a_{24} + \lambda b_{24} = 0, \\ (a_{13} + \lambda b_{13}) p + (a_{23} + \lambda b_{23}) q + (a_{33} + \lambda b_{33}) r + a_{34} + \lambda b_{34} = 0.\end{array}\right\} \quad (2)$$

By this substitution the equations assume the following form:

$$\left.\begin{array}{l}(a_{11}x + a_{12}y + a_{13}z + a_{14}) + \lambda (b_{11}x + b_{12}y + b_{13}z + b_{14}) + k(x-p) = 0, \\ (a_{12}x + a_{22}y + a_{23}z + a_{24}) + \lambda (b_{12}x + b_{22}y + b_{23}z + b_{24}) + k(y-q) = 0, \\ (a_{13}x + a_{23}y + a_{33}z + a_{34}) + \lambda (b_{13}x + b_{23}y + b_{33}z + b_{34}) + k(z-r) = 0,\end{array}\right\} (3)$$

or written shorter

$$\left.\begin{array}{l}A_1 + B_1 \lambda + k (x - p) = 0, \\ A_2 + B_2 \lambda + k (y - q) = 0, \\ A_3 + B_3 \lambda + k (z - r) = 0.\end{array}\right\} \quad \ldots \ldots \quad (4)$$

The surface S_9 is obtained by eliminating p, q, r, k, λ out of the equations (2) and (4).

4. This elimination leads to extensive calculations as the variables appear also as products two by two. We shall here point out the

general course by which at the same time the application in special cases is rendered possible.

The equations (4) can be written as follows:

$$kp = A_1 + B_1 \lambda + kx,$$
$$kq = A_2 + B_2 \lambda + ky,$$
$$kr = A_3 + B_3 \lambda + kz.$$

Let us multiply each of the equations (2) by k and replace the values kp, kq, kr; we then obtain:

$$(a_{11} + \lambda b_{11})(A_1 + B_1 \lambda + kx) + (a_{12} + \lambda b_{12})(A_2 + B_2 \lambda + ky) + (a_{13} + \lambda b_{13})(A_3 + B_3 \lambda + kz) + ka_{14} + kb_{14}\lambda = 0,$$

or:

$$(A_1 + B_1 \lambda) k + (a_{11} + b_{11} \lambda)(A_1 + B_1 \lambda) + (a_{12} + b_{12} \lambda)(A_2 + B_2 \lambda) + (a_{13} + b_{13} \lambda)(A_3 + B_3 \lambda) = 0.$$

We likewise find:

$$(A_2 + B_2 \lambda) k + (a_{12} + b_{12} \lambda)(A_1 + B_1 \lambda) + (a_{22} + b_{22} \lambda)(A_2 + B_2 \lambda) + (a_{23} + b_{23} \lambda)(A_3 + B_3 \lambda) = 0,$$

and finally:

$$(A_3 + B_3 \lambda) k + (a_{13} + b_{13} \lambda)(A_1 + \lambda B_1) + (a_{23} + b_{23} \lambda)(A_2 + B_2 \lambda) + (a_{33} + b_{33} \lambda)(A_3 + B_3 \lambda) = 0. \qquad (5)$$

If we reduce these equations and if we regard k and λ as variables, we shall get as result three quadratic equations, out of which k and λ can be eliminated. As however these equations are linear in k, the elimination of k can take place without any difficulty. By putting the values of k in the first and second equations equal to those in the third and the fourth we deduce from (5):

$$(a_{11} + b_{11} \lambda)(A_1 + B_1 \lambda)(A_2 + B_2 \lambda) + (a_{12} + b_{12} \lambda (A_2 + B_2 \lambda)^2 + (a_{13} + b_{13} \lambda)(A_3 + B_3 \lambda)(A_2 + B_2 \lambda) = (a_{12} + b_{12} \lambda)(A_1 + B_1 \lambda)^2 + (a_{22} + b_{22} \lambda)(A_2 + B_2 \lambda)(A_1 + B_1 \lambda) + (a_{23} + b_{23} \lambda)(A_3 + B_3 \lambda)(A_1 + B_1 \lambda)$$

and

$$(a_{12} + b_{12} \lambda)(A_1 + B_1 \lambda)(A_3 + B_3 \lambda) + (a_{22} + b_{22} \lambda)(A_2 + B_2 \lambda)(A_3 + B_3 \lambda) + (a_{23} + b_{23} \lambda)(A_3 + B_3 \lambda)^2 = (a_{13} + b_{13} \lambda)(A_1 + B_1 \lambda)(A_2 + B_2 \lambda) + (a_{23} + b_{23} \lambda)(A_2 + B_2 \lambda)^2 + (a_{33} + b_{33} \lambda)(A_3 + B_3 \lambda)(A_2 + B_2 \lambda). \qquad (6)$$

When reduced these equations prove to be of order three in λ; we can write them in an abridged form:

$$\left.\begin{aligned} M\lambda^3 + N\lambda^2 + P\lambda + Q = 0, \\ M'\lambda^3 + N'\lambda^2 + P'\lambda + Q' = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which give, according to the method of BEZOUT, the following resultant:

$$\begin{vmatrix} (MN') & (MP') & (MQ') \\ (MP') & (MQ') + (NP') & (NQ') \\ (MQ') & (NQ') & (PQ') \end{vmatrix} = 0. \quad . \quad . \quad (8)$$

5. From this is evident that the form of the final equation is found, but it is a very intricate one, as is proved from the values of the coefficients, given here:

$M = b_{11} B_1 B_2 + b_{12} B_2^2 + b_{13} B_2 B_3 - b_{12} B_1^2 - b_{22} B_1 B_2 - b_{23} B_1 B_3;$

$N = a_{11} B_1 B_2 + b_{11} A_1 B_2 + b_{11} A_2 B_1 + 2b_{12} A_2 B_2 + a_{12} B_2^2 +$
$a_{13} B_2 B_3 + b_{13} A_3 B_2 + b_{13} A_2 B_3 - a_{12} B_1^2 - 2b_{12} A_1 B_1 -$
$a_{22} B_1 B_2 - b_{22} A_2 B_1 - b_{22} A_1 B_2 - a_{23} B_1 B_3 - b_{23} A_3 B_1 -$
$b_{23} A_1 B_3;$

$P = b_{11} A_1 A_2 + a_{11} B_1 A_2 + a_{11} A_1 B_2 + 2a_{12} A_2 B_2 + b_{12} A_2^2 +$
$b_{13} A_2 A_3 + a_{13} A_2 B_3 + a_{13} A_3 B_2 - b_{12} A_1^2 - 2a_{12} A_1 B_1 -$
$b_{22} A_1 A_2 - a_{22} A_1 B_2 - a_{22} A_2 B_1 - b_{23} A_1 A_3 - a_{23} A_1 B_3 -$
$a_{23} A_3 B_1;$

$Q = a_{11} A_1 A_2 + a_{12} A_2^2 + a_{13} A_2 A_3 - a_{12} A_1^2 - a_{22} A_1 A_2 - a_{23} A_1 A_3.$

$M' = b_{12} B_1 B_3 + b_{22} B_2 B_3 + b_{23} B_3^2 - b_{13} B_1 B_2 - b_{23} B_2^2 - b_{33} B_2 B_3;$

$N' = a_{12} B_1 B_3 + b_{12} A_1 B_3 + b_{12} A_3 B_1 + a_{22} B_2 B_3 + b_{22} A_2 B_3 +$
$b_{22} A_3 B_2 + a_{23} B_3^2 + 2b_{23} A_3 B_3 - a_{13} B_1 B_2 - b_{13} A_1 B_2 -$
$b_{13} A_2 B_1 - a_{23} B_2^2 - 2b_{23} A_2 B_2 - a_{33} B_2 B_3 - b_{33} A_3 B_2 -$
$b_{33} A_2 B_3;$

$P' = b_{12} A_1 A_3 + a_{12} A_3 B_1 + a_{12} A_1 B_3 + b_{22} A_2 A_3 + a_{22} A_3 B_2 +$
$a_{22} A_2 B_3 + 2a_{23} A_3 B_3 + b_{23} A_3^2 - b_{13} A_1 A_2 - a_{13} A_1 B_2 -$
$a_{13} B_1 A_2 - 2a_{23} A_2 B_2 - b_{23} A_2^2 - b_{33} A_2 A_3 - a_{33} A_2 B_3 -$
$a_{33} A_3 B_2;$

$Q' = a_{12} A_1 A_3 + a_{22} A_2 A_3 + a_{23} A_3^2 - a_{13} A_1 A_2 - a_{23} A_2^2 - a_{33} A_2 A_3.$

6. With the aid of these expressions the equation of the locus of the axes can be determined for each separate case, which was the purpose of this paper; we shall conclude by giving a few observations.

a. Even in the general case abridgement is possible in the operation. If we assume that the axes of coordinates coincide with the principal axes of one of the surfaces, e. g. of $B = 0$, then $b_{12} = 0$, $b_{13} = 0$,

$b_{23} = 0$, $b_{14} = 0$, $b_{24} = 0$, $b_{34} = 0$, whilst also B_1, B_2, B_3 assume simple forms and all coefficients except Q and Q', are simplified.

At the same time this substitution shows that in equation (8) a factor may be omitted; if namely we make use of the above named values for b_{ik}, we shall find :

$$M = (b_{11} - b_{22}) B_1 B_2 \qquad M' = (b_{22} - b_{33}) B_2 B_3 \,;$$

from this ensues that the first column of the determinant (8) is divisible by B_2. This divisibility is connected with the fact that the equation of the locus of the axes must become of order nine, whilst when developed the determinant (8) becomes of order twelve. So when a complete operation is executed factors must disappear out of (8).

b. Out of the former geometric treatment it is evident, that in some cases the locus of the axes S_q breaks up. As one of the special cases appearing there the case of a circular base curve of the pencil was treated where S_q broke up into a cubic surface and into a surface of order six. The equations of the algebraic treatment of this case become, when one chooses the plane $X\,O\,Y$ as the plane that is intersected according to a pencil of circles:

$$A \equiv a_{11} x^2 + a_{11} y^2 + 2a_{13} xz + 2a_{23} yz + 2a_{34} z + a_{44} = 0\,,$$
$$B \equiv b_{33} z^2 + 2b_{13} xz + 2b_{23} yz + 2b_{14} x + 2b_{34} z = 0.$$

From these equations the simplified values for $M, N \ldots.$ can be deduced.

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