Citation:

J. de Vries, A group of algebraic complexes of rays, in:
KNAW, Proceedings, 7, 1904-1905, Amsterdam, 1905, pp. 627-631

Mathematics. — "*A group of algebraic complexes of rays*". By Prof. Jan de Vries.

§ 1. Supposing the rays a of a pencil (A, α) to be projective to the curves b^n of order n, passing through n^2 fixed points, B_k, of the plane β, we shall regard the complex of the rays resting on homologous lines. For $n = 1$ we evidently find the *tetrahedral complex*.

Out of any point P we project (A, α) on β in a pencil (A', β), generating with the pencil (b^n) a curve c^{n+1}. So we have a *complex of order* $(n+1)$.

Evidently the curve c^{n+1} does not change when the point P is moved along the right line AA'; so the intersections of the ∞^3 cones of the complex (P) with the plane β belong to a system ∞^2. It is easy to see that they form a *net*.

For, if such a curve c^{n+1} is to contain the point X and if b^n_X is the curve through B_k and X, and a_X the ray conjugate to it through A, the point A' must be situated on the right line connecting X with the trace of a_X on the plane β. In like manner a second point through which c^{n+1} must pass, gives a second right line containing A'. The curve c^{n+1} being determined as soon as A' is found, *one* curve c^{n+1} can be brought through two arbitrary points of β.

On the right line $\alpha\beta$ the given pencils determine a $(1, n)$-correspondence; its $(n+1)$ coincidences C_k are situated on each c^{n+1}. So the net has $(n^2 + n + 1)$ fixed base-points [1]).

§ 2. When A' moves along a right line a' situated in β and cutting the plane α in S, the curve c^{n+1} will always have to pass through the n points D_k which a' has in common with the curve b^n conjugate to the ray AS. It then passes through $(n+1)^2$ fixed points, so it describes a pencil comprised in the net.

To the $3n^2$ nodes of curves belonging to that pencil must be counted the n points of intersection of $\alpha\beta$ with that c^n passing through the points B_k and D_k. Hence a' contains, besides S, $(3n^2 - n)$ points A' for which the corresponding curve c^{n+1} possesses a node.

If A' coincides with one of the base-points B_k then the projective pencils (A') and (b^n) generate a c^{n+1} possessing in that point B a node. According to a well known property B is equivalent to two of the nodes appearing in the pencil (c^{n+1}) which is formed

1) To determine this particular net one can choose arbitrarily but $\frac{1}{2}n(n+3) - 1$ points B and three points C.

when A' is made to move along a right line a' drawn through B.

From this ensues in connection with the preceding:

The locus of the vertices of cones of complex possessing a nodal edge is a cone Δ *of order* $n\ (3n - 1)$ *having* A *as vertex and passing twice through each edge* AB_k.

§ 3. If P moves along the plane α then the cone of the complex (P) consists of the plane α and a cone of order n cut by α along the right lines AC_k. So α is a *principal plane* and at the same time part of the singular surface.

The plane β belongs to this too. For, if P lies in β then the rays connecting P with the points of the ray a corresponding to the curve b^n drawn through P belong to the complex. All the remaining rays of the complex through P lie in β. So β is an *n-fold principal plane* and *the singular surface* consists of a simple plane, an n-fold plane and a cone Δ of order $n\ (3n - 1)$.

The complex possesses $(n^2 + n + 2)$ single *principal points,* namely the point A, the n^2 points B_k and the $(n + 1)$ points C_k.

§ 4. The nodes of curves c^p belonging to a net lie as is known on a curve H of order $3\ (p - 1)$ the Hessian of the net, passing twice through each base-point of the net. This property can be demonstrated in the following way.

We assume arbitrarily a right line l and a point M. The c^p touching l in L, cuts ML in $(p - 1)$ points Q more. As the curves passing through M form a pencil, so that $2\ (p - 1)$ of them touch l, the locus of Q passes $2\ (p - 1)$ times through M; so it is of order $3\ (p - 1)$. Through each of its points of intersection S with l one c^p passes having with each of the right lines l and MS two points in common coinciding in S; so S is a node of this c^p.

Consequently the locus of the nodes is a curve of order $3\ (p - 1)$.

If l passes through a base point B of the net then the pencil determined by M cuts in on l an involution of order $(p - 1)$. This furnishing $2\ (p - 2)$ coincidences L, the locus of Q is now of order $(3p - 5)$ only. So B represents for each right line drawn through that point two points of intersection with the locus of the nodes, consequently it is a node of that curve.

If l touches in B_1 the curve c_1^p having a node in B_1 and if one chooses M arbitrarily on this curve, then the curves of the pencil determined by M have in B_1 a fixed tangent and B_1 is one of the coincidences of the involution of order $(p - 1)$. The locus of the nodes has now in B_1 three coinciding points in common with l; consequently it has in B_1 the same tangents as c_1^p.

For the net N^{n+1} of the curves c^{n+1} lying in the plane β the locus of the nodes H breaks up into the right line $\alpha\beta$ and a curve of order $(3n-1)$. For, $\alpha\beta$ forms with each curve b^n a degenerated curve c^{n+1}.

The locus of the nodal edges of the cones of the complex is a cone with vertex A of order $(3n-1)$ having the n^2 right lines AB_k as nodal edges.

§ 5. The tangents in the nodes of a net N^p envelop a curve Z of class $3(p-1)(2p-3)$ [1]), the curve of ZEUTHEN. It breaks up for the net N^{n+1} indicated above; for, the tangents to the curves b^n in their points of intersection with $\alpha\beta$ envelop a curve, which must be a part of the curve Z. The pencil (b^n) is projective to the pencil of its polar curves p^{n-1} with respect to a point O; the points of intersection of homologous curves form a curve of order $(2n-1)$; in each of its points of intersection S with $\alpha\beta$ a curve b^n is touched by OS; so these tangents envelop a curve Z' of class $(2n-1)$.

So for N^{n+1} the curve of ZEUTHEN consists of the envelope Z' and a curve Z'' of class $3n(2n-1)-(2n-1)=(3n-1)(2n-1)$.

The pairs of tangents in the nodes of the genuine curves of N^{n+1} determine on a right line l a symmetric correspondence with characteristic number $(2n-1)(3n-1)$. To the coincidences belong the points of intersection S of l with the curve H; to such a point S are conjugated $(2n-1)(3n-1)-2$ points distinct from S; so S is a double coincidence. The remaining $4(n-1)(3n-1)$ coincidences evidently originate from cuspidal tangents.

The locus of the vertices of cones of the complex, possessing a cuspidal edge consists of $4(n-1)(3n-1)$ edges of the cone Δ.

A general net of order $(n+1)$ contains $12(n-1)n$ cuspidal curves, thus $4(n-1)$ more; therefore each of the $2(n-1)$ figures consisting of the right line $\alpha\beta$ and a curve b^n touching it is equivalent to two curves c^{n+1} with cusp. Evidently the nodes of these figures form with the point C_n the section of $\alpha\beta$ with the curve H.

§ 6. On the traces of a plane π with the planes α and β the pencils (a) and (b^n) determine two series of points in $(n, 1)$-correspondence; the envelope of the right lines connecting homologous points is evidently a curve of class $(n+1)$ touching $\alpha\pi$ in its point of intersection with the ray a conjugate to the curve b^n through

[1]) This has been indicated in a remarkable way by Dr. W. BOUWMAN (Ueber den Ort der Berührungspunkte von Strahlenbüscheln und Curvenbüscheln, N. Archief voor Wiskunde, 2nd series, vol. IV, p. 264).

the point $\alpha\beta\pi$, whilst it touches $\beta\pi$ in its points of intersection with the curve b_0^n for which the corresponding ray passes through $\alpha\beta\pi$.

The curve of the complex of the plane π has the right line $\beta\pi$ for n-fold tangent, so it is rational.

If the curve b_0^n touches the intersection $\beta\pi$, then the multiple tangent is at the same time inflectional tangent.

We now pay attention to the tangents r out of the point $S \equiv \alpha\beta$ to the curve b^n corresponding to a. The envelope of these tangents has the right line $\alpha\beta$ as multiple tangent; its points of contact are the $2(n-1)$ coincidences of the involution, determined by the pencil (b^n) on $\alpha\beta$. As S evidently sends out $n(n-1)$ right lines r the indicated envelope is of class $(n-1)(n+2)$.

The planes containing a curve of the complex of which the n-fold tangent is at the same time inflectional tangent envelop a plane curve of class $(n-1)(n+2)$.

§ 7. The curve (π) can break up in three different ways.

First the point $\alpha\beta\pi$ may correspond to itself, so that (π) breaks up into a *pencil* and into a *curve of class n*. This evidently takes place when π passes through one of the principal points C_k.

Secondly the involution on $\beta\pi$ may break up, so that all its groups contain a fixed point; then also a pencil of rays of the complex separates itself. This will take place, when π passes through one of the principal points B_k.

Thirdly the curve π may contain the principal point A. Then the curve b^n corresponding to the ray $a \equiv \alpha\pi$ determines on $\beta\pi$ the vertices of n pencils, whilst also A is the vertex of a pencil. The curve π is then replaced by $(n+1)$ *pencils*.

In a plane through $\alpha\beta$, thus through all principal points C_k, the curve (π) consists of course also of $(n+1)$ *pencils*.

A break up into *two pencils* with a curve of class $(n-1)$ takes place when the plane π contains two principal points B_k or a point B_k and a point C_k.

§ 8. To obtain an analytical representation of the complex we can start from the equations

$$x_3 = 0 \quad , \quad x_1 + \lambda x_2 = 0;$$
$$x_4 = 0 \quad , \quad a_x^n + \lambda b_x^n = 0.$$

Here a_x^n and b_x^n are homogeneous functions of x_1, x_2, x_3, of order n.

For the points of intersection X and Y of a ray of the complex with α and β we find

$$x_1 : p_{13} = x_2 : p_{23} = x_4 : p_{43},$$
$$y_1 : p_{14} = y_2 : p_{24} = y_3 : p_{34}.$$

After substitution, and elimination of λ, we find an equation of the form

$$p_{23}\,(a_1\,p_{14} + a_2\,p_{24} + a_3\,p_{34})^{(n)} = p_{13}\,(b_1\,p_{14} + b_2\,p_{24} + b_3\,p_{34})^{(n)},$$

by which the exponent between brackets reminds us that we must think here of a *symbolical* raising to a power.

If in $p_{k4} = x_k\,y_4 - x_4\,y_k$ we put the coordinate x_4 equal to zero, we find for the intersection of the cone of the complex of Y on β the equation

$$(y_3\,x_2 - y_2\,x_3)\,(a_1\,x_1 + a_2\,x_2 + a_3\,x_3)^{(n)} = (y_3\,x_1 - y_1\,x_3)\,(b_1\,x_1 + b_2\,x_2 + b_3\,x_3)^{(n)},$$

or shorter

$$y_1\,x_3\,b_x^n - y_2\,x_3\,a_x^n + y_3\,(x_2\,a_x^n - x_1\,b_x^n) = 0.$$

This proves anew, that the intersections of the cones of the complex form a net.

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If a net of curves of order n is represented by an equation in homogeneous coordinates

$$y_1\,a_x^n + y_2\,b_x^n + y_3\,c_x^n = 0$$

to the curve indicated by a system of values $y_1 : y_2 : y_3$ is conjugated the point Y having y_1, y_2, y_3 as coordinates and reversely.

A homogeneous linear relation between the parameters y_k then indicates a right line as locus of Y, corresponding to a pencil comprised in the net.

To the Hessian, H, passing through the nodes of the curves belonging to the net, a curve (Y) corresponds of which the order is easy to determine. For, the pencil represented by an arbitrary right line l_Y has $3(n-1)^2$ nodes. So for the order n'' of (Y) we find $n'' = 3(n-1)^2$.

If one of the curves of a pencil has a node in one of the base-points, it is equivalent to two of the $3(n-1)^2$ curves with node belonging to the pencil. Then the image l_Y touches the curve (Y) and reversely.

Let us suppose that the net has b fixed points, then H passes