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ascribed accordingly to the lengths at 0° C. of the points projecting beyond A and D follow from the equations. These equations give $L_{4_0} + L_{5_0} = 16.587$ and $L_{4_0} + L_{5_0} = 23.095$ for Jena and Thüringer glass respectively, and further;

$$L = L_0\ (1 + at + bt^2)$$

$$V = V_0\ (1 + k_1\, t + k_2\, t^2)$$

Jena glass 16^III^ $\begin{cases} a = 7.74.\ 10^{-6}, b = 0.00882.\ 10^{-6} \\ k_1 = 23.21.\ 10^{-6}, k_2 = 0.0265.\ 10^{-6} \end{cases}$

Thüringer glass (n°. 50) $\begin{cases} a = 9.15\ 10^{-6}, b = 0.0119\ 10^{-6} \\ k_1 = 27.45\ 10^{-6}, k_2 = 0.0357\ 10^{-6} \end{cases}$

The value found for Jenaglass 16^{III} differs much from that obtained by WIEBE and BÖTTCHER [2]) and from those obtained afterwards by THIESEN and SCHEEL [3]) for temperatures between 0° and 100°.

Physics. — "*The motion of electrons in metallic bodies,* III." By Prof. H. A. LORENTZ.

(Communicated in the meeting of March 25, 1905).

§ 16. We may now proceed to examine the consequences to which we are led if we assume *two* kinds of free electrons, positive and negative ones. We shall distinguish the quantities relating to these by the indices 1 and 2; e.g. N_1 and N_2 will be the numbers of electrons per unit of volume, m_1 and m_2 their masses, $\frac{3}{2h_1}$ and $\frac{3}{2h_2}$ the mean squares of their velocities. For simplicity's sake, all electrons of the same sign will be supposed to be equal, even if contained in different metals. As to the charges, these will be taken to have the same absolute value for *all* particles, so that

$$e_2 = -e_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Our new assumption makes only a slight difference in the formula for the electric conductivity; we have only to apply to both kinds of electrons the considerations by which we have formerly found the equation (21). Let a homogeneous metallic bar, having the same temperature throughout, be acted on in the direction of its length by an electric force E; then, just as in § 8, we have for each kind of electrons

1) In the original was given
Jena 16^{III} $a = 7.78$ $b = 0.0090$
Thüringer n° 50 $a = 9.10$ $b = 0.0120$.

2) WIEBE und BÖTTCHER. Z. f. Inst. k. 10, pg. 234. 1890.

3) THIESEN und SCHEEL, Wiss. Abt. der Ph. techn. Reichsanstalt. Bd. II S. 129. 1895.

$$v = \frac{4\pi l A e}{3hm} E.$$

The electric current per unit of area of the normal section, is the sum of the currents due to the positive and the negative particles. We may therefore represent it by

$$\left(\frac{4\pi l_1 A_1 e_1^2}{3h_1 m_1} + \frac{4\pi l_2 A_2 e_2^2}{3h_2 m_2}\right) E$$

and we may write for the coefficient of conductivity

$$\sigma = \sigma_1 + \sigma_2, \quad \ldots \ldots \ldots \quad (49)$$

if

$$\sigma_1 = \frac{4\pi\, l_1\, A_1\, e_1^2}{3h_1\, m_1} \quad , \quad \sigma_2 = \frac{4\pi\, l_2\, A_2\, e_2^2}{3h_2\, m_2}$$

or (cf. § 8)

$$\sigma_1 = \sqrt{\frac{2}{3\pi}} \cdot \frac{l_1 N_1 e_1^2 u_1}{\alpha T} \quad , \quad \sigma_2 = \sqrt{\frac{2}{3\pi}} \cdot \frac{l_2 N_2 e_2^2 u_2}{\alpha T} \quad . . (50)$$

These latter quantities may be called the *partial conductivities* due to the two kinds of electrons.

§ 17. In all the other problems that have been treated in the preceding parts of this paper, we now encounter a serious difficulty. If either the nature of the metal or the temperature changes from one section of the circuit to the next, we can still easily conceive a state of things in which there is nowhere a continual increase of positive or negative electric charge; this requires only that the *total* electric current be 0 for every section of an open circuit and that it have the same intensity for every section of a closed one. But, unless we introduce rather artificial hypotheses, it will in general be found impossible to make each partial current, i. e. the current due to each kind of electrons considered by itself, have the same property. The consequence will be that the number of positive as well as that of negative electrons will increase in some places and diminish in others, the change being the same for the two kinds, so that we may speak of an accumulation of "neutral electricity" in some points and of a diminution of the quantity of neutral electricity in others. Now, supposing all observable properties to remain stationary, as indeed they may, we must of necessity suppose that a volume-element of the metal contains at each instant the same number of really free electrons. This may be brought about in two ways. We may in the first place imagine that all electrons above the normal number that are introduced into the element are immediately caught by the metallic atoms and fixed to them, and that, on the other hand, in

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those places from which electrons are carried away by the two currents, the loss is supplied by a new production of free electrons. This hypothesis would imply a state of the circuit that is not, strictly speaking, stationary and which I shall call "quasi-stationary". Moreover, we should be obliged to suppose that the production of free electrons or the accumulation of these particles in the metallic atoms could go on for a considerable length of time without making itself anyways felt.

In the second place we may conceive each element of volume to contain not only free positive and negative electrons, but, in addition to these, a certain number of particles, consisting of a positive and a negative electron combined. Then, the number of free electrons might be kept constant by a decomposition or a building up of such particles and we could arrive at a really stationary state by imagining a diffusion of this "compound electricity" between different parts of the circuit.

§ 18. The mathematical treatment of our problems is much simplified by the introduction of two auxiliary quantities.

In general, in a non-homogeneous part of the circuit, the acceleration X will be composed of the part X_m, represented by (30), and the part $\frac{e}{m}E$, corresponding to the electric force E. The formula (21) for the flow of a swarm of electrons may therefore be replaced by

$$\nu = \frac{2}{3}\pi l \left[\frac{1}{h^2}\left(-\frac{2hA}{m}\frac{dV}{dx} + \frac{2heA}{m}E - \frac{dA}{dx}\right) + 2\frac{A}{h^3}\frac{dh}{dx}\right]. \quad . \quad (51)$$

This will be 0, if the electric force E has a certain particular value, which I shall denote by E and which is given by

$$\mathrm{E} = \frac{1}{e}\frac{dV}{dx} + \frac{m}{2he}\frac{d \log A}{dx} + \frac{m}{e}\frac{d}{dx}\left(\frac{1}{h}\right). \quad . \quad . \quad . \quad . \quad (52)$$

For any other value of the electric force the flow of electrons will be

$$\nu = \frac{4}{3}\pi l \frac{eA}{hm}(E - \mathrm{E}),$$

and if, in order to obtain the corresponding electric current, we multiply this expression by e, we shall find the product of $E - \mathrm{E}$ by the coefficient of conductivity, in so far as it depends on the kind of electrons considered.

Substituting in (52) the value (14) and applying the result to the positive and the negative electrons separately, we find

$$\left.\begin{aligned} E_1 &= \frac{1}{e_1}\frac{dV_1}{dx} + \frac{2}{3}\frac{aT}{e_1}\frac{d \log A_1}{dx} + \frac{4}{3}\frac{a}{e_1}\frac{dT}{dx} \\ E_2 &= \frac{1}{e_2}\frac{dV_2}{dx} + \frac{2}{3}\frac{aT}{e_2}\frac{d \log A_2}{dx} + \frac{4}{3}\frac{a}{e_2}\frac{dT}{dx} \end{aligned}\right\} \quad . \; . \; . \quad (53)$$

If the area of a normal section is again denoted by Σ, the intensities of the partial currents are given by

$$i_1 = \sigma_1 (E - E_1)\,\Sigma, \quad i_2 = \sigma_2 (E - E_2)\,\Sigma, \quad . \; . \; . \quad (54)$$

and that of the total current, on account of (49), by

$$i = i_1 + i_2 = (\sigma E - \sigma_1 E_1 - \sigma_2 E_2)\,\Sigma. \quad . \; . \; . \; . \quad (55)$$

Putting

$$j_1 = \frac{\sigma_1}{\sigma} i \quad , \quad j_2 = \frac{\sigma_2}{\sigma} i,$$

we may also write

$$i_1 = j_1 + \frac{\sigma_1 \sigma_2}{\sigma}(E_2 - E_1)\,\Sigma, \quad i_2 = j_2 + \frac{\sigma_1 \sigma_2}{\sigma}(E_1 - E_2)\,\Sigma.$$

It appears from these formulae that, whenever E_1 differs from E_2, the partial currents i_1 and i_2 will *not* be proportional to the conductivities σ_1 and σ_2.

§ 19. The above results lead immediately to an equation determining the electromotive force F in an open circuit composed of different metals, between which there is a gradual transition (§ 6) and which is kept in all its parts at the same temperature. Let P and Q be the ends of the circuit and let us reckon x along the circuit in the direction from P towards Q.

The condition for a stationary or a quasi-stationary state is got by putting $i = 0$ in (55). Representing the potential by φ, so that

$$E = -\frac{d\varphi}{dx},$$

we get

$$\frac{d\varphi}{dx} = -\frac{\sigma_1}{\sigma} E_1 - \frac{\sigma_2}{\sigma} E_2. \quad . \; . \; . \; . \; . \quad (56)$$

and finally, taking into account the values (53), in which we now have $\frac{dT}{dx} = 0$, and integrating from P to Q,

$$\varphi_Q - \varphi_P = -\frac{1}{e_1}\int_P^Q \frac{\sigma_1}{\sigma}\frac{dV_1}{dx}\,dx - \frac{1}{e_2}\int_P^Q \frac{\sigma_2}{\sigma}\frac{dV_2}{dx}\,dx -$$

$$- \frac{2}{3} \frac{\alpha T}{e_1} \int_P^Q \frac{\sigma_1}{\sigma} \frac{d \log A_1}{dx} dx - \frac{2}{3} \frac{\alpha T}{e_2} \int_P^Q \frac{\sigma_2}{\sigma} \frac{d \log A_2}{dx} dx \quad . \quad . \quad . \quad (57)$$

At the same time the intensities of the partial currents are given by

$$i_1 = \frac{\sigma_1 \sigma_2}{\sigma} (E_2 - E_1) \Sigma \quad , \quad i_2 = \frac{\sigma_1 \sigma_2}{\sigma} (E_1 - E_2) \Sigma .$$

These values, which are equal with opposite signs, will in general vary along the circuit, so that, even in this simple case, we cannot avoid the complications I have pointed out in § 17. Nor can the difficulty be easily overcome. Indeed, we can hardly admit that the state of two pieces of different metal, in contact with each other and kept at a uniform temperature is not truly stationary. If, in order to escape this hypothesis, we have recourse to the considerations I presented at the end of § 17, we must suppose the neutral electricity to be continually built up in some parts of the system and to be decomposed in other parts. The first phenomenon will be accompanied by a production and the second by a consumption of heat. That these effects should take place in a system whose state is stationary and in which there are no differences of temperature, is however in contradiction with the second law of thermodynamics.

The only way out of the difficulty, if we do not wish to confine ourselves to *one* kind of free electrons, seems to be the assumption that there is no accumulation of neutral electricity at all, i. e. that i_1 and i_2 are simultaneously 0. This would require that $E_1 = E_2$, or in virtue of (53)

$$\frac{1}{e_1} \frac{d V_1}{dx} + \frac{2}{3} \frac{\alpha T}{e_1} \frac{d \log A_1}{dx} = \frac{1}{e_2} \frac{d V_2}{dx} + \frac{2}{3} \frac{\alpha T}{e_2} \frac{d \log A_2}{dx} \quad . \quad . \quad (58)$$

Since $e_2 = - e_1$, we might further conclude that

$$\frac{2}{3} \alpha T \frac{d \log (A_1 A_2)}{dx} + \frac{d (V_1 + V_2)}{dx} = 0,$$

which means that

$$\log (A_1 A_2) + \frac{3}{2 \alpha T} (V_1 + V_2) = \psi (T)$$

ought to have the same value in all parts of the circuit. We should therefore have to regard this expression as a function of the temperature, independent of the nature of the metal [1]).

If we suppose the contact of two metals to have no influence on the number of free electrons in their interior, we must understand by A_1 and A_2 in the above equation quantities characteristic for each

[1]) Cf. DRUDE, Annalen der Physik, 1 (1900), p. 591.

metal and having, for a given temperature, determinate values, whether the body be or not in contact with another metal.

By the assumption $E_1 = E_2$, (56) simplifies into

$$\frac{d\varphi}{dx} = - E_1 = - E_2$$

and (57) becomes

$$\varphi_Q - \varphi_P = \frac{1}{e_1}(V_{1P} - V_{1Q}) + \frac{2}{3}\frac{\alpha T}{e_1} log\left(\frac{A_{1P}}{A_{1Q}}\right) =$$

$$= \frac{1}{e_2}(V_{2P} - V_{2Q}) + \frac{2}{3}\frac{\alpha T}{e_2} log\left(\frac{A_{2P}}{A_{2Q}}\right), \quad . \quad . \quad (59)$$

a formula which is easily seen to imply the law of the tension-series.

§ 20. The question now arises, whether, with a view to simplifying the theory of the thermo-electric current, we shall be allowed to consider E_1 and E_2 as equal, not only in the junctions, but also in the homogeneous parts of the circuit, in which the differences of temperature come into play. This seems very improbable. Indeed, supposing for the sake of simplicity V_1 and V_2 to be, for a given metal, independent of T, so that in a homogeneous conductor $\frac{dV_1}{dx} = 0$ and $\frac{dV_2}{dx} = 0$, we find from (53), putting $E_1 = E_2$,

$$\frac{2}{3}\frac{\alpha T}{e_1}\frac{d\,log\,A_1}{dx} + \frac{4}{3}\frac{\alpha}{e_1}\frac{dT}{dx} = \frac{2}{3}\frac{\alpha T}{e_2}\frac{d\,log\,A_2}{dx} + \frac{4}{3}\frac{\alpha}{e_2}\frac{dT}{dx},$$

or, since $e_2 = - e_1$,

$$T\frac{d\,log\,(A_1 A_2)}{dx} = - 4\frac{dT}{dx},$$

which can hardly be true. It would imply that the product $A_1 A_2$ is inversely proportional to the fourth power of the absolute temperature and this would require in its turn, as may be seen by means of (13) and (14), that the product $N_1 N_2$ should be inversely proportional to T itself.

We are therefore forced to admit inequality of E_1 and E_2. Now, it may be shown that, whatever be the difficulties which then arise in other questions, the theory of the *electromotive force* remains nearly as simple as it was before. For an open circuit we have again to put $i = 0$; hence, the formula (56) will still hold, as may be inferred from (55), if we replace E by $-\frac{d\varphi}{dx}$. The equation for the electromotive force becomes therefore

$$F = \varphi_Q - \varphi_P = -\int_P^Q \frac{1}{\sigma} (\sigma_1 \mathrm{E}_1 + \sigma_2 \mathrm{E}_2)\, dx \quad . \quad . \quad . \quad (60)$$

In the case of a closed circuit, which we get by making the points P and Q coincide, we shall integrate (55) along the circuit after having multiplied that equation by $\frac{dx}{\sigma\Sigma}$ and replaced E by $-\frac{d\varphi}{dx}$. The intensity i being everywhere the same, the result takes the form

$$i\int \frac{dx}{\sigma\Sigma} = F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This is the mathematical expression of OHM's law.

§ 21. It must further be noticed that the equation (60) agrees with the law of the thermo-electric series. This may be shown as follows. If we suppose the temperature to be the same throughout a junction, we may easily infer from what has been said in § 19 that the part of the integral corresponding to such a part of the circuit can be represented as the difference of two quantities, which are both functions of the temperature, but of which one depends solely on the nature of the first metal and the other on that of the second. Considering next a homogeneous part of the circuit between two junctions, we may remark that in this E_1 and E_2 have the form $f(T)\frac{dT}{dx}$ and that the rations $\frac{\sigma_1}{\sigma}$ and $\frac{\sigma_2}{\sigma}$ are functions of the temperature. We may therefore write for the corresponding part of (60)

$$\int_{T'}^{T''} \chi(T)\, dT.$$

This integral, which is to be taken between the temperatures T' and T'' of the junctions, may be considered as the difference of the values, for $T = T'$ and $T = T''$, of a certain quantity depending on the nature of the metal.

Combining these results, we see that the electromotive force in a given circuit is entirely determined by the temperatures of the junctions, and that, if there are two of these between the metals I and II, the electromotive force $F_{I,II}$ we have examined in § 10 c may still be represented by an equation of the form

$$F_{I,II} = \zeta_I(T') - \zeta_I(T'') - \zeta_{II}(T') + \zeta_{II}(T''),$$

the function $\zeta_I(T)$ relating to the first, and the function $\zeta_{II}(T)$ to

H. KAMERLINGH ONNES and W. HEUSE. On the measurement of very low temperatures. V. The expansion coefficient of Jena and Thüringer glass between + 16° and − 182° C.

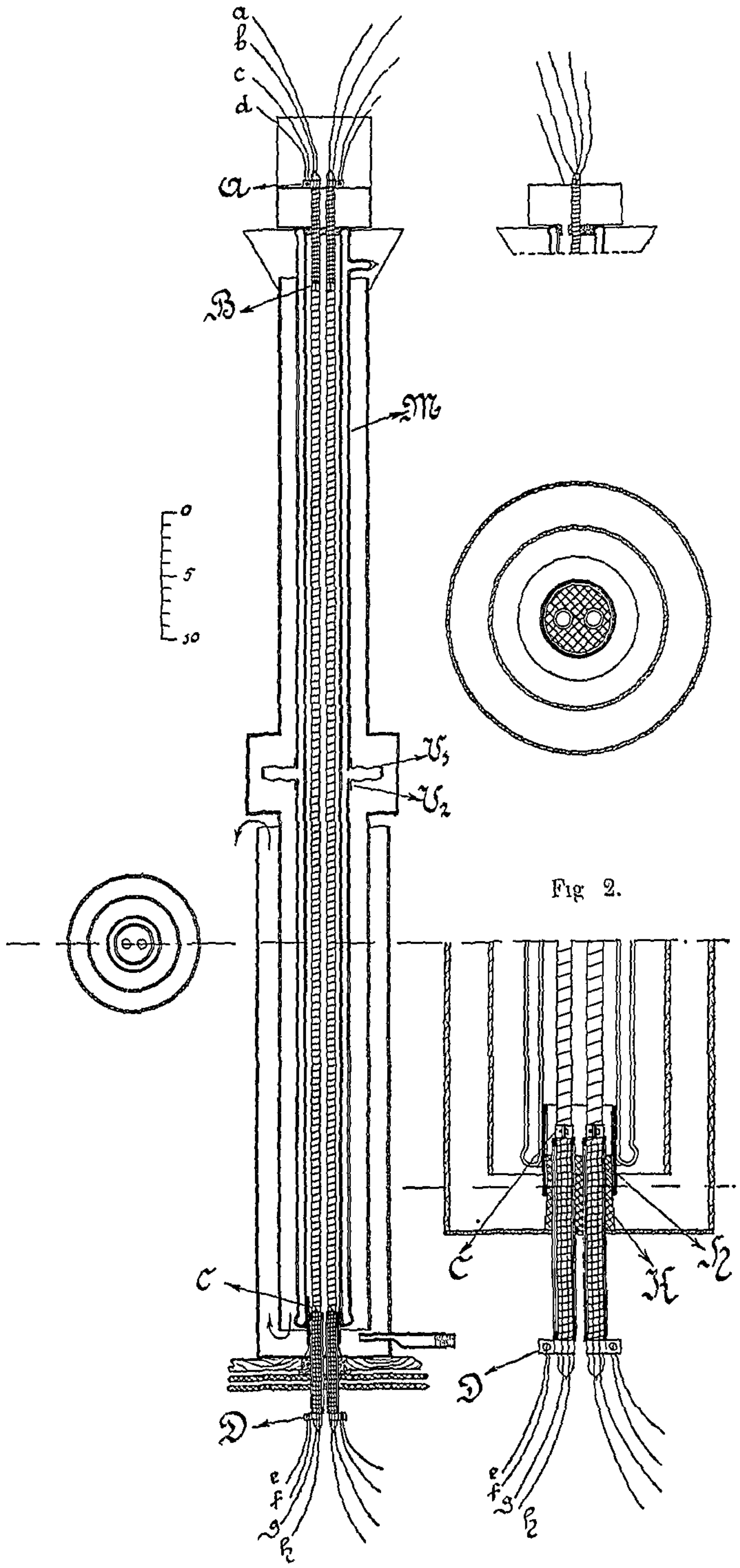

Fig 2.

the second metal. The law of the thermo-electric series may immediately be inferred from this formula. However, in order to obtain this result, it has been necessary to adopt the hypothesis expressed by (58).

I shall terminate this discussion by indicating the way in which our formulae have to be modified, if, in the direction of the circuit, the electrons are acted on not only by the electric force caused by the differences of potential, but also by some other force proportional to their charge and whose line-integral along the circuit is not 0. Let us denote this force, per unit charge, by E_e and let us write for its line-integral

$$\int E_e \, dx = F_e.$$

This latter quantity might be called the "external electromotive force" acting on the circuit. Now, in the formulae (54), we must replace E by $E + E_e$. Consequently, (55) becomes

$$i = \{\sigma (E + E_e) - \sigma_1 \mathrm{E}_1 - \sigma_2 \mathrm{E}_2\},$$

and treating this equation in the same way as we have done (55), we find instead of (61)

$$i \int \frac{dx}{\sigma \Sigma} = F + F_e \, .$$

§ 22. I shall not enter on a discussion of the conduction of heat, the PELTIER-effect and the THOMSON-effect.

In the theory which admits two kinds of free electrons, all questions relating to these phenomena become so complicated that I believe we had better in the first place examine more closely the HALL-effect and allied phenomena. Perhaps it will be found advisable, after all, to confine ourselves to one kind of free electrons, a course in favour of which we may also adduce the results that have been found concerning the masses of the electrons. These tend to show that the positive charges are always fixed to the ponderable atoms, the negative ones only being free in the spaces between the molecules. If however a study of the HALL-effect should prove the necessity of operating with both positive and negative free electrons, we shall be obliged to face all the difficulties attending this assumption.