Citation:

J. de Vries, On linear systems of algebraic plane curves, in:
KNAW, Proceedings, 7, 1904-1905, Amsterdam, 1905, pp. 711-716

which one is recurrent (recurrent br. fig. VI) and leaves the septum to go over into the skin at 1 V and 2 V. fig. V.

The internal branch can be followed up to the vena lateralis (VL fig. IV) and then goes over in a loose plexus. On its way to the vena lateralis the internal branch gives off several filaments, which reach the skin through the intermyotomal septum 3 V—6 V fig. IV and V. Before passing over into the skin these filaments form a loose plexus covering the most ventral part of the myotome.

The roots and mainbranches of the spinal nerve have a submyotomal position and are not bound in their course by the form of the myotome; these branches on the contrary, which go over into the septum to reach the skin, are in their course fixed by the form of the myotome. The final course of the branches in the corium was not traced out with enough accuracy to give results here.

The descriptions given in this note only apply to that region of the trunk which is situated between the thoracic and first dorsal fin.

Conclusions:

I. One single spinal nerve only innervates one single myotome and the intermyotomal tissue through which the nerves pass.

II. The roots and mainbranches of the spinal nerve have a submyotomal position; the branches never perforate a myotome, but run always in the intermyotomal septum to the skin. In general they are to be found between the perimysium and the intermyotomal septum.

III. The spinal nerve shows a primary division into three parts, a posterior, lateral and anterior division in agreement with the differentation of the myotome in a dorsal, lateral and ventral part.

IV. All larger branches are mixed nerves containing elements of the anterior and posterior roots.

Mathematics. — *"On linear systems of algebraic plane curves"*. By Prof. JAN DE VRIES.

§ 1. The points of contact of the tangents out of a point O to the curves c^n of a pencil lie on a curve t^{2n-1} which I shall call the *tangential curve* of O. It is a special case of a curve indicated by CREMONA [1]). By EMIL WEYR [2]), GUCCIA [3]) and W. BOUWMAN [4]) it has been applied when proving the properties of pencils and nets.

1) CREMONA—CURTZE, Einleitung in eine geometrische Theorie der ebenen Curven (1865) p. 119.

2) Sitzungsberichte der Akademie in Wien, LXI, 82.

3) Rendiconti del Circolo matematico di Palermo (1895), IX, 1.

4) Nieuw Archief voor Wiskunde (1900), IV, 258.

If a linear system $(c^n)_k$ of ∞^k curves c^n is given, we can consider the locus of the points P_{k+1}, where a curve of that system has a $(k+1)$-pointed contact with a right line, passing through the fixed point O.

To determine the order $\varphi(k)$ of the locus (P^{k+1}) I consider the curves $(c^n)_k$ having in the points P of the right line l a k-pointed contact with the corresponding right line OP. Each ray OP cuts the curve individualized by P moreover in $(n-k)$ points Q. Each point of intersection of l with the locus of the points Q being evidently a point P_{k+1}, the locus (Q) is a curve of order $\varphi(k)$.

The curves of $(c^n)_k$ passing through O form a system $(c^n)_{k-1}$. The order of the locus of the points P_k where a c^n of this latter system has a k-pointed contact with OP is evidently indicated by $\varphi(k-1)$. So on l lie $\varphi(k-1)$ points P for which one of the corresponding points Q coincides with O; in other words the locus (Q) passes $\varphi(k-1)$ times through O, so it is of order $\varphi(k-1)+(n-k)$.

To determine $\varphi(k)$ we have now the recurrent relation

$$\varphi(k)=\varphi(k-1)+(n-k).$$

From this we deduce

$$\varphi(k)=\varphi(1)+\tfrac{1}{2}(k-1)(2n-k-2).$$

Here $\varphi(1)$ represents the order of the tangential curve, thus $(2n-1)$. So we find

$$\varphi(k)=\tfrac{1}{2}(k+1)(2n-k).$$

The locus of the points where a curve c^n, belonging to a k-fold infinite linear system has a $(k+1)$-pointed contact with a right line passing through a fixed point O is a curve of order $\frac{1}{2}(k+1)(2n-k)$, on which O is a $\frac{1}{2}k(k+1)$-fold point.

For $(c^n)_k$ determines on a right line r through O an involution of order n and rank k. The number of $(k+1)$-fold elements of this involution amounts to $(k+1)(n-k)$; that is at the same time the number of points P_{k+1}, lying on r. Consequently O is an $\frac{1}{2}k(k+1)$-fold point on (P_{k+1}).

§ 2. Each ray r through a fixed point O is touched by $2(n-1)$ curves c^n of a pencil (c^n); the points of contact T are the double points of the involution determined by (c^n) on r. The curves c^n indicated by these points T intersect r moreover in $2(n-1)(n-2)$ points S. When r rotates round O the points S will describe a curve which I shall call the *satellite curve* of O.

This curve passes $(n+1)(n-2)$ times through O; for if r coincides with one of the tangents out of O to c^n passing through

O one of the points S lies in O. *So the curve* (S) *is of order* $(n+1)(n-2)+2(n-1)(n-2)=(n-2)(3n-1)$.

If B is a base-point of (c^n), then only $2(n-2)$ points T (the double points of an I^{n-1}) lie on OB outside O and B. So OB touches in B the tangential curve of O whilst it is $(n-2)$-fold tangent of (S).

Each of the $2(n-2)$ curves c^n touching OB projects a point S in B. So each base-point is a $2(n-2)$-fold point of the satellite curve.

The common points of the tangential curve t^{2n-1} and the satellite curve $s^{(3n-1)(n-2)}$ form four groups.

First there are $(n+1)(n-2)$ united in O.

Secondly $2(n-2)$ lie in each base-point B.

Thirdly the two curves touch each other at each inflectional point sending its tangent through O.

Fourthly they cut each other in the points of contact of each double tangent passing through O.

Now the inflectional tangents of a pencil envelop a curve of class $3n(n-2)$. [1])

So the number of points of contact of inflectional tangents through O amounts to

$$(n-2)(3n-1)(2n-1)-(n-2)(n+1)-2(n-2)n^2-6n(n-2)=$$
$$=4n(n-2)(n-3).$$

The double tangents of the curves c^n *belonging to a pencil envelop a curve of class* $2n(n-2)(n-3)$.

§ 3. Following Emil Weyr [2]) we consider the curve c^{n+1} generated by the pencil (c^n) with the pencil projectively conjugate to it of the tangents in a base-point B. As each c^n cuts its tangent moreover in $(n-2)$ points, B is a threefold point of the c^{n+1}. From this ensues easily that through B can be drawn $(n+4)(n-3)$ tangents to c^{n+1}. As many double tangents of the pencil (c^n) have one of their points of contact in B.

We shall now consider the satellite curve of B. On each ray r through B lie $2(n-2)$ points of contact T, so $2(n-2)(n-3)$ points S. If r coincides with one of the double tangents just mentioned, one of the points S lies in B. So B is an $(n+4)(n-3)$-fold point on (S) and the order of (S) proves to be equal to $(n+4)(n-3)+2(n-2)(n-3)=3n(n-3)$.

The tangential curve of B has in B a threefold point; for a ray

---

[1]) For this is the number of tangents of t^{2n-1} which besides the n^2 tangents of OB can be drawn through O.

[2]) Sitzungsberichte der Akademie in Wien LXI, 82.

through B bears but $(2n-4)$ points T, whilst the curve t is of order $(2n-1)$.

Of the common points of t^{2n-1} and $s^{3n(n-3)}$ there are $3(n+4)(n-3)$ lying in B, $2(n-3)$ in each of the remaining (n^2-1) base-points and two in each of the inflectional points sending their tangent through B.

The number of those inflectional tangents is $3n(n-2)-9$, as each of the three inflectional tangents, having their inflectional point in B, must be counted three times. This is evident when we consider a curve of (c^3), where a base-point can lie only on inflectional tangents for which it is inflectional point itself. This number amounts to *three*, whilst the class of the envelope of the inflectional tangents is *nine*.

So we find for the number of the points of contact, not lying in B, of double tangents out of B

$$3n(n-3)(2n-1)-3(n+4)(n-3)-2(n-3)(n^2-1)-6(n-3)(n+1)=$$
$$=4(n-3)(n-4)(n+1).$$

So B lies on $2(n-4)(n-3)(n+1)$ double tangents. This number is $2(n-3)(n+4)$ less than the number of double tangents out of an arbitrary point. The $(n-3)(n+4)$ double tangents having one of its points of contact in B must thus be counted twice.

The envelope of the double tangents has in each base-point an $(n+4)(n-3)$*-fold point.*

§ 4. The locus of the points of contact D of the double tangents of (c^n) evidently passes $(n+4)(n-3)$-times through each base-point (§ 3). An arbitrary c^n having on its double tangents $n(n-2)(n^2-9)$ points of contact D, the curve D and c^n intersect each other in $n^2(n+4)(n-3)+n(n-2)(n^2-9)$ points. Consequently the locus of the points of contact D is a curve of order $(n-3)(2n^2+5n-6)$. [1])

We shall now consider the locus of the points W in which a c^n is intersected by its double tangents.

As each base-point B lies on $2(n-4)(n-3)(n+1)$ double tangents (§ 3) the curve W passes with as many branches through B. So it has with an arbitrary c^n in common $2n^2(n-4)(n-3)(n+1)+$ $+\frac{1}{2}n(n-2)(n^2-9)(n-4)$ points. From this ensues that the curve (W) is of order $\frac{1}{2}(n-4)(n-3)(5n^2+5n-6)$.

The curves (D) and (W) have outside the base-points a number of points in common equal to

$$\tfrac{1}{2}(n-4)(n-3)^2(5n^2+5n-6)(2n^2+5n-6)-$$
$$-2n^2(n-4)(n-3)^2(n+1)(n+4).$$

1) See P. H. Schoute, Wiskundige opgaven, II, 307.

From this ensues:

In a pencil (c^n)

$$\tfrac{1}{2}(n-4)(n-3)^2(10n^4+35n^3-21n^2-80n+20)$$

curves have an inflectional point of which the tangent touches the curve in one other point more.

§ 5. The locus of the inflectional points I of (c^n) has a threefold point in each base-point and a node in each of the $3(n-1)^2$ nodes of the pencil, out of which we immediately find that the curve (I) is of order $6(n-1)$ and of class $6(n-2)(4n-3)$ [2]).

Let us now deduce the order of the locus of the points V determined by a c^n on its inflectional tangents.

As a base-point B lies on $3(n-3)(n+1)$ inflectional tangents the curve (V) passes with as many branches through B. So with an arbitrary c^n it has $3n^2(n-3)(n+1)+3n(n-2)(n-3)$ points in common.

Consequently (V) is a curve of order $3(n-3)(n^2+2n-2)$. Now the curves (I) and (V) have besides the base-points a number of points in common represented by

$$18(n-1)(n-3)(n^2+2n-2)-9n^2(n-3)(n+1).$$

These points can only have risen from the coincidence of inflectional points with one of the points they have in common with the c^n under consideration, thus from *tangents with fourpointed contact.* Such an *undulation point,* being equivalent to two inflectional points, is point of contact for (I) and (V) from which ensues:

A pencil (c^n) *contains* $\frac{9}{2}(n-3)(n^3+n^2-8n+4)$ *curves with an undulation point.*

§ 6. Let a threefold infinite linear system of curves c^n be given.

The c^n osculating a right line l in the point P cuts the ray OP drawn through the arbitrary point O moreover in $(n-1)$ points Q.

The curves of $(c^n)_3$ passing through O form a net $(c^n)_2$ determining on l the groups of an involution I_2^n. The latter having $3(n-2)$ threefold elements, the locus (Q) passes $3(n-2)$-times through O, so it is of order $(4n-7)$.

Each of its points of intersection K with l is evidently a node on a curve of $(c^n)_3$, with l and OK for tangents.

Each right line is nodal tangent for $(4n-7)$ *curves of the system.*

From this ensues that the locus of the nodes K sending one of

[2]) See Bobek, Casopis (Prague), XI, 283.

their tangents through the point M chosen arbitrarily is a curve of order $(4n - 5)$; for M is a node of a c^n, so it lies on two branches of (K).

Each point K of the arbitrary right line l is a node of a curve belonging to $(c^n)_3$. The points of intersection M and M' of the tangents in K with the right line m chosen arbitrarily are pairs of a symmetric correspondence with characteristic number $(4n - 5)$. To the coincidences belongs the point of intersection M_0 of l and m, and twice even, because the c^n, having in that point a node, furnishes two points M_0' coinciding with M_0. The remaining coincidences originate from tangents in cusps. From this ensues:

The locus of the cusps of a threefold infinite linear system of curves of order n is a curve of order $4\,(2n - 3)$.

Mathematics. — *"Some characteristic numbers of an algebraic surface."* By Prof. JAN DE VRIES.

In the following paper we shall show how by easy reasoning we can find an amount of the characteristic numbers of a general surface of order n [1]. To this end we shall make use of scrolls formed by principal tangents or double tangents.

§ 1. First I consider the scroll $\mathbf{A}$ of the principal tangents a of which the points of contact A lie in a given plane α. The curve α^n along which α cuts the surface Φ^n is evidently nodal curve of $\mathbf{A}$. The tangents in the $3n\,(n - 2)$ inflectional points of α^n being principal tangents of Φ^n, the scroll $\mathbf{A}$ has $3n\,(n - 2)$ right lines and the curve α^n to be counted twice in common with Φ^n, so it is a scroll of order $n\,(3n - 4)$.

The two principal tangents a and a' in a point of α^n have each three points in common with Φ^n; consequently α^n belongs six times to the section of $\mathbf{A}$ and Φ^n. These surfaces have moreover a twisted curve of order $n^2\,(3n-4)-6n$ in common containing the $3n(n-2)\,(n-3)$ points where Φ^n is cut by the principal tangents a situated in α. In each of the remaining $n\,(11n - 24)$ points of intersection of this curve with α the surface Φ^n has *four* coinciding points of intersection in common with a. From this ensues:

The locus of the points in which Φ^n *possesses a fourpointed tangent* (flecnodal line) *is a twisted curve of order* $n\,(11n-24)$.

[1]) We find the indicated numbers in SALMON-FIEDLER, "Analytische Geometrie des Raumes", dritte Auflage, II, p. 622—644, and in SCHUBERT, "Kalkül der abzählenden Geometrie", p. 236.