their tangents through the point M chosen arbitrarily is a curve of order $(4n - 5)$; for M is a node of a c^n, so it lies on two branches of (K).

Each point K of the arbitrary right line l is a node of a curve belonging to $(c^n)_3$. The points of intersection M and M' of the tangents in K with the right line m chosen arbitrarily are pairs of a symmetric correspondence with characteristic number $(4n - 5)$. To the coincidences belongs the point of intersection M_0 of l and m, and twice even, because the c^n, having in that point a node, furnishes two points M_0' coinciding with M_0. The remaining coincidences originate from tangents in cusps. From this ensues:

The locus of the cusps of a threefold infinite linear system of curves of order n is a curve of order $4\ (2n - 3)$.

Mathematics. — *"Some characteristic numbers of an algebraic surface."* By Prof. JAN DE VRIES.

In the following paper we shall show how by easy reasoning we can find an amount of the characteristic numbers of a general surface of order n [1]). To this end we shall make use of scrolls formed by principal tangents or double tangents.

§ 1. First I consider the scroll $\mathbf{A}$ of the principal tangents a of which the points of contact A lie in a given plane α. The curve α^n along which α cuts the surface Φ^n is evidently nodal curve of $\mathbf{A}$. The tangents in the $3n\,(n - 2)$ inflectional points of α^n being principal tangents of Φ^n, the scroll $\mathbf{A}$ has $3n\,(n - 2)$ right lines and the curve α^n to be counted twice in common with Φ^n, so it is a scroll of order $n\,(3n - 4)$.

The two principal tangents a and a' in a point of α^n have each three points in common with Φ^n; consequently α^n belongs six times to the section of $\mathbf{A}$ and Φ^n. These surfaces have moreover a twisted curve of order $n^2\,(3n - 4) - 6n$ in common containing the $3n(n - 2)\,(n - 3)$ points where Φ^n is cut by the principal tangents a situated in α. In each of the remaining $n\,(11n - 24)$ points of intersection of this curve with α the surface Φ^n has *four* coinciding points of intersection in common with a. From this ensues:

The locus of the points in which Φ^n possesses a fourpointed tangent (flecnodal line) *is a twisted curve of order* $n\,(11n - 24)$.

1) We find the indicated numbers in SALMON-FIEDLER, "Analytische Geometrie des Raumes", dritte Auflage, II, p. 622—644, and in SCHUBERT, "Kalkül der abzählenden Geometrie", p. 236.

§ 2. I now determine the order of the scroll **B** formed by the principal tangents cutting ϕ^n in points B of the plane β.

Out of each point B of the section β^n start $(n-3)(n^2+2)$ principal tangents; this number indicates at the same time the number of sheets of **B** which cut each other along β^n. The inflectional tangents lying in β^n evidently belong $(n-3)$-times to the indicated scroll. So its order is equal to

$$n(n-3)(n^2+2)+3n(n-2)(n-3)=n(n-1)(n-3)(n+4).$$

According to § 1 $n(3n^2-4n-6)$ principal tangents have their point of contact A on α^n and one of their points of intersection B on β^n. So this number indicates the order of the curve along which ϕ^n is osculated by **B**. Beside this curve of contact and the manyfold curve β^n the surfaces ϕ^n and **B** have still in common the locus of the points B' which determine the principal tangents AB moreover on ϕ^n. This curve (B') is of order $n^2(n-1)(n-3)(n+4)-$ $-3n(3n^2-4n-6)-n(n-3)(n^2+2)=n(n-2)(n-4)(n^2+5n+3)$.

§ 3. To find how often the point A coincides with one of the $(n-4)$ points B', I shall project the pairs of points (A, B') out of a right line l. The planes through l are arranged in this way in a correspondence with the characteristic numbers $n(3n^2-4n-6)(n-4)$ and $n(n-2)(n-4)(n^2+5n+3)$. Each right line a resting on l evidently contains $(n-4)$ pairs (A, B'), so it furnishes an $(n-4)$-fold coincidence. The remaining coincidences originate from coincidences $A\equiv B'$. Now $n(3n^2-4n-6)(n-4)+n(n-2)(n-4)(n^2+5n+3)-$ $-n(n-1)(n-3)(n+4)(n-4)=n(n-4)(6n^2+2n-24)$. So this is the number of fourpointed tangents which cut ϕ^n in a point B of β^n.

The points of intersection of ϕ^n with its fourpointed tangents form a curve of order $2n(n-4)(3n^2+n-12)$.

If f is the order of the scroll of the fourpointed tangents then it is evident that we have the relation

$$nf=4n(11n-24)+2n(n-4)(3n^2+n-12)=2n^2(n-3)(3n-2).$$

The fourpointed tangents form a scroll of order $2n(n-3)(3n-2)$.

If we make the point of contact F of a fourpointed tangent to correspond to the $(n-4)$ points G which that tangent has still in common with ϕ^n, a system of pairs of points (F, G) is formed, of which the number of coincidences can be determined again with the aid of the correspondence in which they arrange the planes through an axis l. By the way indicated above we find for this number:

$$n(11n-24)(n-4)+2n(n-4)(3n^2+n-12)-2n(n-3)(3n-2)(n-4)=n(n-4)(35n-60).$$

The surface ϕ^n possesses $5n(n-4)(7n-12)$ fivepointed tangents.

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§ 4. Returning to the scroll **B** (§ 2) I consider the points of intersection of the twisted curve (B') with the plane β. Each point of intersection of α^n with an inflectional tangent lying in β can be regarded as the point B, each one of the remaining $(n-4)$ as a point B'. Hence the curve (B') meets $3n(n-2)(n-3)(n-4)$-times β^n on the inflectional tangents of β^n. In each of the remaining points of intersection of (B') with β we find that φ^n is touched by a right line having elsewhere three coinciding points in common with φ^n. Such a right line is called by me a tangent $t_{2,3}$, A being its *point of osculation*, B its *point of contact.*

The points of contact of the tangents $t_{2,3}$ form a curve of order $n(n-2)(n-4)(n^2+2n+12)$.

§ 5. In each point C of the curve γ^n according to which φ^n is cut by the plane γ I shall regard the $(n-3)(n+2)$ tangents c which touch φ^n moreover in a point C'. On the scroll **C** of the double tangents c the curve γ^n is a manyfold curve in which $(n-3)(n+2)$ sheets meet. Each double tangent situated in γ representing two right lines of **C** the order of this scroll is equal to

$$n(n-3)(n+2)+n(n-2)(n-3)(n+3) \text{ or } n(n-3)(n^2+2n-4).$$

The surfaces φ^n and **C** touch each other along the locus (C') of the two points of contact. Of this curve the plane γ contains the points of contact of the right lines c lying in γ besides the points $C \equiv C'$, where a right line c is a fourpointed tangent. So the order of (C') is $n(n-2)(n^2-9)+n(11n-24)$ or $n(n^3-2n^2+2n-6)$.

Besides the curve (C') to be counted twice and the curve γ^n to be counted $2(n-3)(n+2)$-times **C** and φ^n have moreover in common the locus of the points S determined by the double tangents c on φ^n. The curve (S) is of order $n^2(n-3)(n^2+2n-4)-2n(n^3-2n^2+2n-6)-2n(n-3)(n+2)$ or $n(n-4)(n^3+n^2-4n-6)$.

To the points of (S) lying in γ belong the points of intersection of γ^n with its double tangents c. As each of the two points of contact of c can be regarded as point C these points of intersection S must be counted twice. The remaining $n(n-4)(n^3+n^2-4n-6)-n(n-2)(n^2-9)(n-4)$ points S lying in γ are apparently points of osculation of the tangents $t_{2,3}$. So from this ensues:

The points of osculation of the principal tangents touching φ^n moreover elsewhere form a curve of order $n(n-4)(3n^2+5n-24)$.

The curves (A) and (B) formed by the points of osculation and the points of contact of the tangents $t_{2,3}$ have the points of contact of the fivepointed tangents in common. Taking this into account we find (by again projecting out of an axis l) for the order of the scroll of the right lines $t_{2,3}$ the expression $n(n-2)(n-4)(n^2+2n+12)+$ $+n(n-4)(3n^2+5n-24)-5n(n-4)(7n-12)$.

The principal tangents of φ^n *which moreover touch the surface form a scroll of order* $n(n-3)(n-4)(n^2+6n-4)$.

§ 6. The double tangents c cutting φ^n in points D of the plane δ form a scroll **D**, on which the section δ^n of φ^n with δ is a manyfold curve bearing $\frac{1}{2}(n-3)(n-4)(n^2+n+12)$ [1]) sheets. As moreover every double tangent of δ^n belongs to $(n-4)$ different points D the order of **D** is equal to

$$\tfrac{1}{2}n(n-3)(n-4)(n^2+n+2)+\tfrac{1}{2}n(n-2)(n-3)(n+3)(n-4)= n(n-1)(n+2)(n-3)(n-4).$$

According to § 5 $n(n-4)(n^3+n^2-4n-6)$ double tangents c have one of their points of contact C in a given plane γ and at the same time one of their points of contact D in the plane δ. So this number indicates the order of the curve along which **D** and φ^n touch each other. If we take the manyfold curve δ^n into consideration, it is evident that the points D' which the right lines of **D** have in common with φ^n besides the points of contact C and the points of intersection D lying in δ, form a twisted curve (D') the order of which in equal to

$$n^2(n-1)(n+2)(n-3)(n-4)-2n(n-4)(n^3+n^2-4n-6)-$$
$$\tfrac{1}{2}n(n-3)(n-4)(n^2+n+2)=\tfrac{1}{2}n(n-2)(n-4)(n-5)(2n^2+5n+3).$$

This curve evidently cuts δ $(n-4)(n-5)$-times on each double tangent of δ^n. In each of its remaining points of intersection with δ the surface φ^n is touched by a right line, which is tangent to the surface in two more points. From this ensues:

The points of contact C of the threefold tangents of φ^n *form a curve (C) of order* $\frac{1}{2}n(n-2)(n-4)(n-5)(n^2+5n+12)$.

§ 7. On each right line c of the scroll **D** lie $(n-5)$ points D' which can be arranged in $\frac{1}{2}(n-5)(n-6)$ pairs D', D''. If these pairs of points are projected out of an axis l by pairs of planes λ', λ'', these form a symmetric system, the characteristic number of which is $\frac{1}{2}n(n-2)$ $(n-4)(n-5)(2n^2+5n+3)(n-6)$. Each right line c cutting l determines a plane λ evidently representing $(n-5)(n-6)$ coincidences $\lambda'\equiv\lambda''$. The remaining coincidences of the system (λ) originate from coincidences $D'\equiv D''$, thus from threefold tangents d. As however

[1]) In Cremona—Curtze, *Theorie der Oberflächen*, page 66 we find the expression $\frac{1}{2}(n-3)(n-4)(n^2+n-2)$ by mistake for the number of double tangents cutting φ^n in one of its points.

each of the three points of contact of a right line d can be formed when D' coincides with D'' the number of threefold tangents cutting Φ^n on the curve σ^n is but the third part of the number of the indicated coincidences of (λ), thus equal to

$$\tfrac{1}{3}\, n\,(n-4)\,(n-5)\,(n-6)\,\{(n-2)\,(2n^2+5n+3)-(n-1)\,(n+2)\,(n-3)\}= \tfrac{1}{3}\, n\,(n-4)\,(n-5)\,(n-6)\,(n^3+3n^2-2n-12).$$

This is at the same time the order of the curve (D) formed by the points D which the threefold tangents d have still in common with Φ^n.

Now we can also find the order x of the scroll (d). This scroll being touched by Φ^n in the points of (C) and being cut in the points (D) we have namely

$$nx = n\,(n-2)\,(n-4)\,(n-5)\,(n^2+5n+12) + \tfrac{1}{3}\, n\,(n-4)\,(n-5)\,(n-6)\,(n^3+3n^2-2n-12).$$

Out of this we find

The threefold tangents of Φ^n form a scroll the order of which is $\tfrac{1}{3}\, n\,(n-3)\,(n-4)\,(n-5)\,(n^2+3n-2)$ [1]).

§ 8. To find the degree of the spinodal curve I consider the pairs of principal tangents a, a' of which the common point of contact A lies in the plane α. If two rays s and s' of a pencil (S, σ) are conjugate to each other, when they rest on two right lines a and a', then in (S, σ) a symmetric correspondence with characteristic number $n\,(3n-4)$ is formed. The coincidences can be brought to three groups.

First a and a' can cut the same ray s; their plane of connection is then tangential plane, their point of intersection A lies on the polar surface of S. Such a ray s coincides with two of the rays s' conjugate to it. So the first group contains $n\,(n-1)$ double coincidences.

Secondly s can cut the curve α^n; then too it coincides with two rays s'. So the second group consists of n double coincidences.

Finally a single coincidence is formed when a coincides with a'. The number of these coincidences evidently amounts to $2n\,(3n-4) - 2n\,(n-1) - 2n = 4n\,(n-2)$. From this ensues:

The parabolic points form a twisted curve (spinodal line) *of order* $4n\,(n-2)$.

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[1]) In Salmon-Fiedler we find on page 638 by mistake n^2+3n+2 instead of n^2+3n-2.

On page 643 we find the derivation of the number of fourfold tangents and of the numbers of tangents $t_{1,2}$, $t_{3,2,2}$ and $t_{3,3}$.