Mathematics. "*The equation of order nine representing the locus of the principal axes of a pencil of quadratic surfaces.* By Mr. K. Bes. (Communicated by Prof. J. Cardinaal).

1. In These Proceedings of Jan. 28th 1905 appears a communication by Prof. Cardinaal: "On the equations by which the locus of the principal axes of a pencil of quadratic surfaces is determined."

2. Prof. Cardinaal deduces three non-homogeneous equations of order two between two variable parameters λ and k, and tries to arrive at the equation of the demanded surface by elimination of these parameters. The result obtained by him (8) seems to be an equation of order 12. This is incongruent with the result arrived at geometrically, which made an equation of order nine to be expected. This incongruency is attributed to factors, which the equation arrived at may contain, but these factors are not indicated.

3. The method of elimination described in my paper "Théorie générale de l'élimination" (Verhandelingen, Vol. VI, n°. 7) gives the means to set aside this incongruency and to determine in reality the equation sought for by Prof. Cardinaal.

To this end we can start from his equations (5) after having made them homogeneous with respect to the variable parameters, which may be done by assuming the equation (1) of the pencil of surfaces in the form:

$$\mu A + \lambda B = 0 .$$

If now we develop the equations (5), they assume the following form:

$$\left.\begin{aligned}
&(a_{11}A_1+a_{12}A_2+a_{13}A_3)\mu^2+(a_{11}B_1+a_{12}B_2+a_{13}B_3+b_{11}A_1+b_{12}A_2+b_{13}A_3)\lambda\mu+\\
&\quad+(b_{11}B_1+b_{12}B_2+b_{13}B_3)\lambda^2+A_1\mu k+A_1\lambda k=0\,,\\
&(a_{12}A_1+a_{22}A_2+a_{23}A_3)\mu^2+(a_{12}B_1+a_{22}B_2+a_{23}B_3+b_{12}A_1+b_{22}A_2+b_{23}A_3)\lambda\mu+\\
&\quad+b_{12}B_1+b_{22}B_2+b_{23}B_3)\lambda^2+A_2\mu k+B_2\lambda k=0\,,\\
&(a_{13}A_1+a_{23}A_2+a_{33}A_3)\mu^2+(a_{13}B_1+a_{23}B_3+a_{33}B_3+b_{13}A_1+b_{23}A_2+b_{33}A_3)\lambda\mu+\\
&\quad+(b_{13}B_1+b_{23}B_2+b_{33}B_3)\lambda^2+A_3\mu k+B_3\lambda k=0\,.
\end{aligned}\right\}(a)$$

The coefficients of these equations are linear functions of the variable coordinates x, y and z. To simplify we can introduce the following notations:

$$\begin{aligned}
P_1 &= a_{11}A_1 + a_{12}A_2 + a_{13}A_3\,,\\
P_2 &= a_{12}A_1 + a_{22}A_2 + a_{23}A_3\,,\\
P_3 &= a_{13}A_1 + a_{23}A_2 + a_{33}A_3\,,\\
Q_1 &= a_{11}B_1 + a_{12}B_2 + a_{13}B_3 + b_{11}A_1 + b_{12}A_2 + b_{13}A_3\,,\\
Q_2 &= a_{12}B_1 + a_{22}B_2 + a_{23}B_3 + b_{12}A_1 + b_{22}A_2 + b_{23}A_3\,,\\
Q_3 &= a_{13}B_1 + a_{23}B_2 + a_{33}B_3 + b_{13}A_1 + b_{23}A_2 + b_{33}A_3\,,\\
R_1 &= b_{11}B_1 + b_{12}B_2 + b_{13}B_3\,,\\
R_2 &= b_{12}B_1 + b_{22}B_2 + b_{23}B_3\,,\\
R_3 &= b_{13}B_1 + b_{23}B_2 + b_{33}B_3\,,
\end{aligned}$$

by which the equations (a) pass into the following:

$$\left.\begin{array}{l} P_1\mu^2 + Q_1\lambda\mu + R_1\lambda^2 + A_1\mu k + B_1\lambda k = 0\,, \\ P_2\mu^2 + Q_2\lambda\mu + R_2\lambda^2 + A_2\mu k + B_2\lambda k = 0\,, \\ P_3\mu^2 + Q_3\lambda\mu + R_3\lambda^2 + A_3\mu k + B_3\lambda k = 0\,, \end{array}\right\} \quad . \quad . \quad . \quad (b).$$

4. Which condition now must exist between the coefficients of these equations if they are to allow of a mutual system of roots? The answer is that no condition is demanded for this. These equations are namely satisfied independent of the value of the coefficients by the system of roots:

$$\lambda = 0\,, \quad \mu = 0\,, \quad k \text{ arbitrary.}$$

The result arrived at by applying the method indicated in § 118 of my paper. "Théorie générale de l'élimination" agrees with this. According to this method we should have to find for the resultant the quotient of two determinants successively of order 15 and of order 3. In the case under consideration where we have

$$a_6 = 0\,, \quad b_6 = 0 \text{ and } c_6 = 0\,,$$

we always obtain, in whatener way we choose the determinants, as quotient a, quantity which is identically zero.

So the above-mentioned equation (8) can be nothing else but an identity.

5. This result having been fixed it is no longer difficult to answer the question how to obtain the equation of the demanded locus. To this end we must express the condition that the equations (b) are satisfied by a second system of roots.

The condition in demand is, that all determinants are equal to zero contained in the assemblant (85) appearing in § 118 of the already mentioned paper. Applied to the equations (b) it gives but one equation, namely

$$\left|\begin{array}{ccccccccc} P_1 & & & P_2 & & & P_3 & & \\ Q_1 & P_1 & & Q_2 & P_2 & & Q_3 & P_3 & \\ R_1 & & P_1 & R_2 & & P_2 & R_3 & & P_3 \\ A_1 & Q_1 & & A_2 & Q_2 & & A_3 & Q_3 & \\ B_1 & R_1 & Q_1 & B_2 & R_2 & Q_2 & B_3 & R_3 & Q_3 \\ & & R_1 & & & R_2 & & & R_3 \\ & A_1 & & & A_2 & & & A_3 & \\ & B_1 & A_1 & & B_2 & A_2 & & B_3 & A_3 \\ & & B_1 & & & B_2 & & & B_3 \end{array}\right| = 0\,,$$

this being the equation of the demanded locus. It is of order nine agreeing to the geometrical researches of Prof. CARDINAAL.