Astronomy. — *"Approximate formulae of a high degree of accuracy for the relations of the triangles in the determination of an elliptic orbit from three observations."* By J. WEEDER. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN.)

The places in space occupied by the observed planet or comet at the instants t_1, t_2 and t_3 are indicated by P_1, P_2 and P_3, the position of the sun is indicated by Z.

For the determination of an elliptic orbit we mainly proceed as follows: first by means of successive approximation we derive the distances $P_1Z = r_1$, $P_2Z = r_2$, $P_3Z = r_3$ from the data of the observations, from which distances the elements of the orbit are directly computed without using the intervals of time. From the obtained ellipse we can again derive the intervals of time in order to test the accuracy of the results and compare them with the real ones. In case they perfectly agree, the ellipse found satisfies all the conditions of the problem, but as a rule this is not so. The cause of it is that, in order to calculate the distances r_1, r_2, and r_3, we use approximate formulae to express the relations $\frac{\text{triangle } P_1ZP_2}{\text{triangle } P_1ZP_3} = n_3$ and $\frac{\text{triangle } P_2ZP_3}{\text{triangle } P_1ZP_3} = n_1$ in terms of the intervals of time and of the three distances to be found, while neglecting the terms of the 2nd, 3rd or 4th order with respect to the intervals. Indeed, different expressions have been proposed for n_1 and n_3, some recommending themselves by greater simplicity, others by greater accuracy, but, so far as I know, in the general case of unequal intervals none of them contain the quantities of the fourth order with respect to the intervals.

The errors in the calculated distances r_1, r_2 and r_3 and those in the elements of the orbit derived from them are generally of the same order as that of the terms omitted in the expressions for n_1 and n_3.

Accurate and at the same time simple expressions for n_1 and n_3 have been given by J. W. GIBBS[1]).

The purpose of this paper is to develop, according to GIBBS' method, expressions for n_1 and n_3 which include the terms of the 4th order; at the same time a new derivation of GIBBS' relations is given.

In the ellipse sought let P be the position of the heavenly body at the time t, x and y its heliocentric rectangular coordinates in the

[1]) J. W. GIBBS: On the determination of elliptic orbits from three complete observations. Memoirs of the national academy of sciences. Vol. IV, 2; p. 81. Washington 1889.

plane of the orbit, and $r = ZP$, then x and y satisfy the following differential equations

$$\frac{d^2x}{d\tau^2} = -\frac{x}{r^3} = \ddot{x} \qquad \frac{d^2y}{d\tau^2} = -\frac{y}{r^3} = \ddot{y},$$

wherein we have put $\tau = k(t - t_1)$ as independent variable instead of the time t; τ is therefore the time reckoned from the epoch of the first observation and expressed in the unit for which, in the solar system, the acceleration $= 1$ at a distance from the sun which is adopted as unit of length; k is the constant of GAUSS [$log\, k = 8.235\ 581\ 4414 - 10$].

While designating the rectangular coordinates of P_1, P_2, P_3 by corresponding indices I remark that $n = \dfrac{x_1 y - y_1 x}{x_1 y_3 - y_1 x_3} = \dfrac{\text{triangle } P_1 ZP}{\text{triangle } P_1 ZP_3}$ satisfies a similar differential equation as x and y, namely:

$$\frac{d^2n}{d\tau^2} = -\frac{n}{r^3} = \ddot{n}.$$

At the times	$t_1\ (\tau = 0)$,	$t_2\ (\tau = \tau_3)$,	$t_3\ (\tau = \tau_2)$
the values of n are	0	$+ n_3$	$+ 1$ and
the values of $\ddot{n}$	0	$-\dfrac{n_3}{r_2^{\,3}}$	$-\dfrac{1}{r_3^{\,3}}$

Consequently in the development of n in a series of ascending powers of τ after MAC LAURIN, the terms of the power zero and 2 will be wanting. If in this expansion we do not go farther than the 4th power of τ, we require only 3 indefinite coefficients which may be eliminated from the following 4 relations:

$$\begin{aligned} n_3 &= K_1\tau_3 + K_3\tau_3^{\,3} + K_4\tau_3^{\,4} + f_4 \\ 1 &= K_1\tau_2 + K_3\tau_2^{\,3} + K_4\tau_2^{\,4} + F_4 \\ -\frac{n_3}{r_2^{\,3}} &= \quad + 6K_3\tau_3 + 12K_4\tau_3^{\,2} + f_2 \\ -\frac{1}{r_3^{\,3}} &= \quad + 6K_3\tau_2 + 12K_4\tau_2^{\,2} + F_2 . \end{aligned}$$

The remaining relation yields an expression for n_3 in τ_2, τ_3, r_2, r_3 and the remainders f_4, F_4, f_2 and F_2.

The indices which I have used for the remainders, indicate the order of these terms with respect to τ; F_4, for instance, which begins with $K_5\tau_2^{\,5}$ is of the 4th order of τ, which is evident when we express the coefficients K in terms of the derivatives of n for $\tau = 0$ and develop the latter by means of the differential equation for n as products of $\dot{n}_0$. For clearness I shall here give this development:

$$\frac{d^3n}{d\tau^3} = -n\dot{z} - z\dot{n}\ , \quad \text{where } z \text{ is put for } \frac{1}{r^3}$$

$$\frac{d^4n}{d\tau^4} = (z^2 - \ddot{z})\, n - 2\dot{z}\dot{n}$$

$$\frac{d^5n}{d\tau^5} = (4z\dot{z} - \dddot{z})\, n + (z^2 - 3\ddot{z})\, \dot{n}\ .$$

From the differential equation $\frac{d}{d\tau}(r^3\ddot{r} + r) = 0$, satisfied by r, we can derive the following differential equation for $z = \frac{1}{r^3}$:

$$\dddot{z} = \dot{z}\left(5\frac{\ddot{z}}{z} - \frac{40}{9}\frac{\dot{z}^2}{z^2} - z\right)$$

which may serve to eliminate $\dddot{z}$ from the higher derivatives.

$$\frac{d^5n}{d\tau^5} = 5\left(z - \frac{\ddot{z}}{z} + \frac{8}{9}\frac{\dot{z}^2}{z^2}\right)\dot{z}n + (z^2 - 3\ddot{z})\,\dot{n}\ .$$

For $\tau = 0$, n is equal to zero and $\dot{n} = K_1$, hence

$$K_3 = -\frac{1}{6}z_1K_1 \quad , \quad K_4 = -\frac{1}{12}\dot{z}_1K_1 \quad , \quad K_5 = \frac{z_1^{\,2} - 3\ddot{z}_1}{120}K_1\ .$$

If we substitute the expressions for the coefficients K in the second of the 4 relations, this becomes:

$$1 = K_1\tau_2\left\{1 - \frac{1}{6}z_1\tau_2^{\,2} - \frac{1}{12}\dot{z}_1\tau_2^{\,3} + \frac{z_1^{\,2} - 3\ddot{z}_1}{120}\tau_2^{\,4}\ldots\right\}$$

and from this it clearly appears that K_1 and the other coefficients K, in so far as they depend on the intervals, are of the order $\frac{1}{\tau_2}$.

From the 4 relations with the indefinite coefficients K_1, K_3, K_4 we find by eliminating the latter:

$$\left\{n_3 - f_4 - \frac{\tau_2^{\,2} + \tau_2\tau_3 - \tau_3^{\,2}}{12}\left(\frac{n_3}{r_2^{\,3}} + f_2\right)\right\}\tau_2 = \left\{1 - F_4 - \frac{\tau_3^{\,2} + \tau_3\tau_2 - \tau_2^{\,2}}{12}\left(\frac{1}{r_3^{\,3}} + F_2\right)\right\}\tau_3\ .$$

From this equation I solve:

$$n_3 = \frac{\tau_3}{\tau_2}\,\frac{1 - \dfrac{\tau_3^{\,2} + \tau_3\tau_2 - \tau_2^{\,2}}{12r_3^{\,3}}}{1 - \dfrac{\tau_2^{\,2} + \tau_2\tau_3 - \tau_3^{\,2}}{12r_2^{\,3}}} + R_4$$

where

$$R_4 = \frac{f_4 + f_2 \dfrac{\tau_2^2+\tau_2\tau_3-\tau_3^2}{12} - \dfrac{\tau_3}{\tau_2}\left\{F_4 + F_2 \dfrac{\tau_3^2+\tau_3\tau_2-\tau_2^2}{12}\right\}}{1 - \dfrac{\tau_2^2+\tau_2\tau_3-\tau_3^2}{12 r_2^3}}.$$

This remainder is apparently of the 4th order with respect to the intervals. If we neglect the terms of higher order than the fourth we can replace in R_4: f_4 by $K_5\tau_3^5$, F_4 by $K_5\tau_2^5$, f_2 by $20K_5\tau_3^3$ and F_2 by $20K_5\tau_2^3$; and we obtain as supplementary term, accurate to the fourth order

$$+\frac{1}{3} K_5\tau_3 (\tau_2 + \tau_3)(\tau_2 - \tau_3)(2\tau_2 - \tau_3)(\tau_2 - 2\tau_3),$$

which expression vanishes on account of the last factor, in the case of equal intervals.

The corresponding approximate formula for n_1 can be derived by developing the relation $\frac{\text{triangle } PZP_3}{\text{triangle } P_1ZP_3}$, depending on the time, in ascending powers of $k(t_3 - t)$ and further by proceeding in the same manner as we have done for n_3. The result for n_1 is derived from the preceding result by interchanging the indices 1 and 3, in which case τ_1 stands for $k(t_3 - t_2)$, hence:

$$n_1 = \frac{\tau_1}{\tau_2} \frac{1 - \dfrac{\tau_1^2+\tau_1\tau_2-\tau_2^2}{12 r_1^3}}{1 - \dfrac{\tau_2^2+\tau_2\tau_1-\tau_1^2}{12 r_2^3}} - R_4.$$

The remainder of n_1 is not only of the same order as that of n_3, but even in the 4th order it has the same absolute value, with a different sign however. This appears clearly when, using the relation $\tau_2 = \tau_1 + \tau_3$, we express the correction of the 4th order for n_3 in terms of τ_1 and τ_3; this correction takes the following form, which is symmetrical with respect to τ_1 and τ_3:

$$\frac{1}{3} K_5\tau_1\tau_3 (2\tau_1 + \tau_3)(\tau_1 + 2\tau_3)(\tau_1 - \tau_3).$$

In the remainder of n_1 the coefficient L_5 may be assumed equal to K_5. Therefore these approximate formulae always give for $n_1 + n_3$ an accurate value (comp. p. 758), including the terms of the 4th order of the interval.

The denominators of these expressions for n_1 and n_3, although here different in form, are indeed identical; the expressions themselves agree with those derived from the fundamental equation adopted by Gibbs between the 3 vectors ZP_1, ZP_2 and ZP_3 which can be easily reduced to the form:

$$\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_1^2+\tau_1\tau_2-\tau_2^2}{12r_1^3}\right)\overline{ZP_1}+\frac{\tau_3}{\tau_2}\left(1-\frac{\tau_3^2+\tau_3\tau_2-\tau_2^2}{12r_3^3}\right)\overline{ZP_3}=$$
$$=\left(1-\frac{\tau_2^2+\tau_2\tau_3-\tau_3^2}{12r_2^3}\right)\overline{ZP_2}.$$

This equation is satisfied by the real places of the object when we neglect a residual of the 5th order with respect to the intervals of time. This signifies little, however, when compared with the accuracy of the places calculated after GIBBS' method, which rigorously satisfy them; for each set of vector corrections $\Delta\overline{ZP_1}$, $\Delta\overline{ZP_2}$ and $\Delta\overline{ZP}$ does not lessen the agreement below the 5th order with respect to the intervals of time, provided they satisfy the condition

$$\frac{\tau_1}{\tau_2}\Delta\overline{ZP_1}+\frac{\tau_3}{\tau_2}\Delta\overline{ZP_3}=\Delta\overline{ZP_2}$$

and are not below the 3rd order with respect to those intervals.

Because in GIBBS' method the relations n_1 and n_3 contain errors of the 4th order, it would follow from this that the places computed after this method are inaccurate in the 4th order also. But thanks to the circumstance that GIBBS' method includes for n_1+n_3 the terms of the 4th order in all cases, its results are yet correct in terms of the 4th order.

This special feature of GIBBS' method has been pointed out by E. WEISS [1]).

In order to obtain for n_1 and n_3 expressions including in all cases the 4th order of the intervals of time and containing besides them only $\frac{1}{r_1^3}=z_1$, $\frac{1}{r_2^3}=z_2$ and $\frac{1}{r_3^3}=z_3$, I have used the relation derived on p. 754 $K_3=-\frac{1}{6}z_1K_1$.

Starting from the development

$$n=K_1\tau+K_3\tau^3+K_4\tau^4+K_5\tau^5+\text{remainder of the 5th order}$$

I can make use of the following relations between the coefficients K, the quantities z_1, z_2, z_3 and n_3.

$$\begin{aligned} n_3 &= K_1\tau_3+K_3\tau_3^3+K_4\tau_3^4+K_5\tau_3^5+f_5\\ 1 &= K_1\tau_2+K_3\tau_2^3+K_4\tau_2^4+K_5\tau_2^5+F_5\\ -z_3n_3 &= \quad +6K_3\tau_3+12K_4\tau_3^2+20K_5\tau_3^3+f_3\\ -z_2 \quad &= \quad +6K_3\tau_2+12K_4\tau_2^2+20K_5\tau_2^3+F_3\\ 0 &= K_1z_1+6K_3 \end{aligned}$$

[1]) E. WEISS, Ueber die Bestimmung der Bahn eines Himmelskörpers aus drei Beobachtungen. Denkschriften der Mathem. Naturw. Classe der Wiener Akademie. Bd. LX (1893).

By eliminating K_1, K_3, K_4 and K_5 we derive from them the following equation:

$$\tau_2 \left\{ (n_3 - f_5)\left(1 + \frac{\tau_2^2(2\tau_2 - 5\tau_3)}{60\tau_3} z_1\right) - \right.$$

$$\left. - (n_3 z_2 + f_3)\left(\frac{2\tau_2^3 + 2\tau_2^2\tau_3 + 2\tau_2\tau_3^2 - 3\tau_3^3}{60\tau_3} - \frac{\tau_2^2\tau_3(4\tau_2 - 3\tau_3)}{720} z_1\right)\right\} =$$

$$= \tau_3 \left\{ (1 - F_5)\left(1 + \frac{\tau_3^2(2\tau_3 - 5\tau_2)}{60\tau_2} z_1\right) - \right.$$

$$\left. - (z_3 + F_3)\left(\frac{2\tau_3^3 + 2\tau_3^2\tau_2 + 2\tau_3\tau_2^2 - 3\tau_2^3}{60\tau_2} - \frac{\tau_3^2\tau_2(4\tau_3 - 3\tau_2)}{720} z_1\right)\right\}.$$

For shortness I replace the expressions which only depend on the intervals of time by single letters, putting

$$A_{23} = \frac{\tau_2^2(2\tau_2 - 5\tau_3)}{60\tau_3} \qquad A_{32} = \frac{\tau_3^2(2\tau_3 - 5\tau_2)}{60\tau_2}$$

$$B_{23} = \frac{-2\tau_2^3 - 2\tau_2^2\tau_3 - 2\tau_2\tau_3^2 + 3\tau_3^3}{60\tau_3} \qquad B_{32} = \frac{-2\tau_3^3 - 2\tau_3^2\tau_2 - 2\tau_3\tau_2^2 + 3\tau_2^3}{60\tau_2}$$

$$C_{23} = \frac{\tau_2^2\tau_3(4\tau_2 - 3\tau_3)}{720} \qquad C_{3.2} = \frac{\tau_3^2\tau_2(4\tau_3 - 3\tau_2)}{720}.$$

then the equation, solved with respect to n_3, yields for this relation the following expression:

$$n_3 = \frac{\tau_3}{\tau_2} \times \frac{1 + A_{32} z_1 + B_{3.2} z_3 + C_{32} z_1 z_3}{1 + A_{23} z_1 + B_{23} z_2 + C_{23} z_1 z_2} + R_5 \quad . \quad . \quad (I)$$

The remainder R_5 contains the quantities F_5, f_5, F_3 and f_3; for these I set, in order to form the value of R_5 in the 5th order with respect to the intervals of time, $f_5 = K_6 \tau_3^6$, $F_5 = K_6 \tau_2^6$, $f_3 = 30 K_6 \tau_3^4$ and $F_3 = 30 K_6 \tau_2^4$; I then find:

$$R_5 = \tfrac{1}{2} K_6 (\tau_2^4 - \tau_2^3\tau_3 - \tau_2^2\tau_3^2 - \tau_2\tau_3^3 + \tau_3^4)\tau_3(\tau_2 - \tau_3).$$

As the root of the 4th power equation $1 - x - x^2 - x^3 + x^4 = 0$ lies between zero and 1, viz. $x = 0{,}5806$, the terms of the 5th order will vanish from the residual, if $\tau_3 = 0.5806\,\tau_2$.

We obtain the corresponding approximation for n_1 when we derive an expression from that for n_3 by interchanging everywhere the indices 1 and 3, hence:

$$n_1 = \frac{\tau_1}{\tau_2} \times \frac{1 + A_{12} z_3 + B_{12} z_1 + C_{1.2} z_3 z_1}{1 + A_{2.1} z_3 + B_{2.1} z_2 + C_{2.1} z_3 z_2} \quad . \quad . \quad . \quad (II)$$

The meaning of the new letters agrees with the rules for the interchange of the indices 1 and 3.

$$A_{1.2} = \frac{\tau_1^2\,(2\tau_1 - 5\tau_2)}{60\,\tau_2} \qquad A_{2.1} = \frac{\tau_2^2\,(2\tau_2 - 5\tau_1)}{60\,\tau_1}$$

$$B_{1.2} = \frac{-2\tau_1^3 - 2\tau_1^2\tau_2 - 2\tau_1\tau_2^2 + 3\tau_2^3}{60\,\tau_2} \qquad B_{2.1} = \frac{-2\tau_2^3 - 2\tau_2^2\tau_1 - 2\tau_2\tau_1^2 + 3\tau_1^3}{60\,\tau_1}$$

$$C_{1.2} = \frac{\tau_1^2\,\tau_2\,(4\tau_1 - 3\tau_2)}{720} \qquad C_{2.1} = \frac{\tau_2^2\,\tau_1\,(4\tau_2 - 3\tau_1)}{720}.$$

In the remainder which belongs to this expression for n_1, the term of the 5th order:

$$\tfrac{1}{2} L_6 \,(\tau_2^4 - \tau_2^3\tau_1 - \tau_2^2\tau_1^2 - \tau_2\tau_1^3 + \tau_1^4)\,\tau_1\,(\tau_2 - \tau_1)$$

will vanish if $\tau_1 = 0.5806\,\tau_2$, therefore the term can never vanish at the same time for n_1 and for n_3.

L_6 occurs as coefficient of τ_6 in the development of $\dfrac{\text{triangle } PZP_3}{\text{triangle } P_1ZP_3}$ in ascending powers of $\tau = k\,(t_3 - t)$, while K_6 indicates the coefficient of τ^6 in the development of $\dfrac{\text{triangle } P_1ZP}{\text{triangle } P_1ZP_3}$, where the variable τ means $k\,(t - t_1)$.

If the first of these developments were performed in powers of $k\,(t - t_3) = -\tau$, there would exist between each pair of corresponding coefficients a relation implying that its sum with regard to τ_2 would be of one order higher than the coefficients themselves. Therefore, neglecting terms of higher order than the 5th, we may assume that the coefficients K_6 and L_6 are identical in absolute value, yet differ in sign.

Of a similar relation I have made use on p. 755, where in the remainders of the 4th order I assumed the coefficients identical. In the new expressions for n_1 and n_3 we can now, by putting $L_6 = -K_6$, derive the following value for the remainder of the 5th order of $n_1 + n_3$:

$$\tfrac{1}{2} K_6\,\tau_1\tau_2\tau_3\,(\tau_1 - \tau_3)\,(2\tau_2^2 + \tau_1\tau_3).$$

Therefore when the intervals of time are equal, the error in $n_1 + n_3$ is of the 6th order.

If according to the indicated method we include the terms of the 4th order, we find for the 3rd relation $\dfrac{\text{triangle } P_2ZP_1}{\text{triangle } P_3ZP_2} = \dfrac{n_3}{n_1}$

$$\frac{n_3}{n_1} = \frac{\tau_3}{\tau_1} \times \frac{1 + A_{3.1}z_2 + B_{3.1}z_3 + C_{3.1}z_2z_3}{1 + A_{1.3}z_2 + B_{1.3}z_1 + C_{1.3}z_2z_1}, \quad \ldots \quad (III)$$

and with it as remainder of the 5th order

$$+ \tfrac{1}{2} K_6 \frac{\tau_3}{\tau_1}\,\tau_2^2(\tau_3^4 + \tau_3^3\tau_1 - \tau_3^2\tau_1^2 + \tau_3\tau_1^3 + \tau_1^4)$$

From one of the examples from GAUSS' *Theoria Motus* (Libr. II, Sect. I cc. 156—158) I have computed the 3 relations according to the formulae I, II and III. The rigorously correct values of those relations and the results of GIBBS' expressions for this example I borrow from P. HARZER's *Bestimmung und Verbesserung der Bahnen von Himmelskörpern nach drei Beobachtungen* p. 8. [1])

The heliocentric motion of the planet Pallas was from the 1st to the 3d observation 22°33′.

$$\log \tau_1 = 9.8362703 \quad \log \tau_2 = 0.0854631 \quad \log \tau_3 = 9.7255594$$
$$\log r_1 = 0.3630906 \quad \log r_2 = 0.3507163 \quad \log r_3 = 0.3369508$$

These values for $\log r$ are also taken from HARZER and differ a little from those according to GAUSS.

Results for	$\log n_1$	and for	$\log n_3$
GIBBS	9.7572961	GIBBS	9.6480108
formula II	9.7572928	formula I	9.6480167
rigorous	9.7572923	rigorous	9.6480201.

Formula III yields: $\log \dfrac{n_3}{n_1} = 9.8907237.$

With the given logarithms agree the following values:

	n_1		n_3		$n_3 : n_1$
rigorous	0.5718634		0.4446518		0.7775491
f. II	0.5718641	f. I	0.4446484	f. III	0.7775418
differences	— 0.0000007		+ 0.0000034		+ 0.0000073

From the expressions given for the remainders of the 5th order I calculated that they are in the ratio of — 9, + 72 and + 140. If we compare these numbers with the residuals, it appears that for our example they would vanish to the 7th decimal if we succeeded in including also the terms of the 5th order in the expressions.

As to the calculation of the quantities A and B dependent on τ_1, τ_2, and τ_3, I remark that it may be performed quickly if we modify these forms in the following way:

$$A_{12} = -\frac{\tau_3}{10}\left(\frac{1}{2}\frac{\tau_1^2}{\tau_3}+\frac{1}{3}\frac{\tau_1^2}{\tau_2}\right) \quad B_{1.2} = +\frac{\tau_1}{10}\left\{\frac{1}{2}\frac{\tau_2^2}{\tau_1}-\frac{1}{3}\left(\tau_1+\tau_2+\frac{\tau_1^2}{\tau_2}\right)\right\}$$

$$A_{2.1} = -\frac{\tau_3}{10}\left(\frac{1}{2}\frac{\tau_2^2}{\tau_3}-\frac{1}{3}\frac{\tau_2^2}{\tau_1}\right) \quad B_{2.1} = +\frac{\tau_2}{10}\left\{\frac{1}{2}\frac{\tau_1^2}{\tau_2}-\frac{1}{3}\left(\tau_1+\tau_2+\frac{\tau_2^2}{\tau_1}\right)\right\}$$

$$A_{32} = -\frac{\tau_1}{10}\left(\frac{1}{2}\frac{\tau_3^2}{\tau_1}+\frac{1}{3}\frac{\tau_3^2}{\tau_2}\right) \quad B_{3.2} = +\frac{\tau_3}{10}\left\{\frac{1}{2}\frac{\tau_2^2}{\tau_3}-\frac{1}{3}\left(\tau_2+\tau_3+\frac{\tau_3^2}{\tau_2}\right)\right\}$$

$$A_{2.3} = -\frac{\tau_1}{10}\left(\frac{1}{2}\frac{\tau_2^2}{\tau_1}-\frac{1}{3}\frac{\tau_2^2}{\tau_3}\right) \quad B_{2.3} = +\frac{\tau_2}{10}\left\{\frac{1}{2}\frac{\tau_3^2}{\tau_2}-\frac{1}{3}\left(\tau_2+\tau_3+\frac{\tau_2^2}{\tau_3}\right)\right\}$$

$$A_{3.1} = -\frac{\tau_2}{10}\left(\frac{1}{2}\frac{\tau_3^2}{\tau_2}+\frac{1}{3}\frac{\tau_3^2}{\tau_1}\right) \quad B_{3.1} = +\frac{\tau_3}{10}\left\{\frac{1}{2}\frac{\tau_1^2}{\tau_3}+\frac{1}{3}\left(\tau_1-\tau_3+\frac{\tau_3^2}{\tau_1}\right)\right\}$$

$$A_{1.3} = -\frac{\tau_2}{10}\left(\frac{1}{2}\frac{\tau_1^2}{\tau_2}+\frac{1}{3}\frac{\tau_1^2}{\tau_3}\right) \quad B_{1.3} = +\frac{\tau_1}{10}\left\{\frac{1}{2}\frac{\tau_3^2}{\tau_1}+\frac{1}{3}\left(\tau_3-\tau_1+\frac{\tau_1^2}{\tau_3}\right)\right\}$$

[1]) Publication der Sternwarte in Kiel, XI.