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to a far greater depth and when they had been taken out of it, they did not so easily regain their former support.

DURAND (1845) gave the explanation which a large, old seedling on mercury suggested to him. It had stayed so long on it, that an adhesive layer had been formed on the mercury of sufficient thickness to fasten the plant to some extent. That therefore all seedlings whose roots penetrate into mercury, should stick to it by such a layer is not true. The penetration takes places after a short time when the mercury is still bright.

DUTROCHET (1845) accepted DURAND's explanation and made experiments on the formation of the sticky layer. But he did not put to himself the question whether in all the observed cases such a "plaster" had been present.

WIGAND (1854) has undoubtedly obtained PINOT's results. In his discussion however he confused and complicated the question as MULDER had done. For this reason later investigators did not bestow much attention to the paradox which he had so clearly pronounced. Where he speaks of penetration into dry mercury, this must certainly not be taken literally; the soaked seeds retain a layer of water.

HOFMEISTER (1860) studied the penetration of roots in relation with his theory of the plastic apex. He did not obtain the result of PINOT and WIGAND and accepted DURAND's explanation which also DUTROCHET had accepted.

Later investigators all followed HOFMEISTER's opinion.

Mathematics. — "*The harmonic curves belonging to a given plane cubic curve.*" By Prof. JAN DE VRIES.

1. The "harmonic" curve of a given point P with respect to a given plane cubic curve k^3 is the locus of the point H separated harmonically from P by two of the points of intersection A_1, A_2, A_3 of k^3 and PH[1]). We shall determine the equation of the harmonic curve h^3 when k^3 is indicated by the equation

$$a^3_x \equiv b^3_x \equiv (a_1 x_1 + a_2 x_2 + a_3 x_3)^{(3)} = 0,$$

and P by the coordinates (y_1, y_2, y_3).

[1]) This curve appears in STEINER's treatise: "Ueber solche algebraische Curven, welche einen Mittelpunkt haben, ..." (J. of Crelle, XLVII), and is there more generally specified as a curve of order n. Stereometrically it has been determined by Dr. H. DE VRIES in his dissertation: "Over de restdoorsnede van twee volgens eene vlakke kromme perspectivische kegels, en over satellietkrommen", Amsterdam 1901, p. 6 and 88.

To the points of intersection of k^3 and h^3 belong the points of contact of the six tangents from P to k^3. If A_1 is one of the remaining three points of intersection, then A_2 and A_3 are harmonically separated by A_1 and P, that is P lies on the polar conic of A_1; from this follows however that A_1 lies on the polar line of P. So the curve h^3 passes through the points of intersection of k^3 with both the polar conic p^2 and the polar line p^1 of P. Its equation is therefore of the form $\mu a^3_x + a_y a^2_x b^2_y b_x = 0$. If point X belongs to the harmonic curve of point Y, it is evident that Y lies on the harmonic curve of X; so our equation must be symmetric with regard to the variables x_i and y_i; that is, it has the form

$$a^3_x b^3_y + \lambda a^2_x a_y b_x b^2_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To determine λ we suppose P to be lying on $x_3=0$ and we then consider the points of h^3 which are lying on $x_3=0$. If we represent the linear factors of the binary form $a^3_x \equiv b^3_x \equiv (a_1 x_1 + a_2 x_2)^{(3)}$ by p_x, q_x and r_x, then the points H_1, H_2, H_3 are indicated by the equation

$$h^3_x \equiv (p_x q_y + p_y q_x)(p_x r_y + p_y r_x)(q_x r_y + q_y r_x) = 0,$$

or by

$$h^3_x \equiv \sum_6 p^2_x q_x q_y r^2_y + 2 p_x p_y q_x q_y r_x r_y = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We now have

$$3 a^2_x a_y \equiv p_x q_x r_y + p_x q_y r_x + p_y q_x r_x,$$

$$3 b_x b^2_y \equiv p_x q_y r_y + p_y q_x r_y + p_y q_y r_x,$$

and as we moreover have

$$p_y q_y r_y \equiv b^3_y,$$

we find out of (2)

$$h^3_x \equiv 9 a^2_x a_y b_x b^2_y - a^3_x b^3_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation also represents the harmonic curve, if we but again regard a^3_x as the symbol for $(a_1 x_1 + a_2 x_2 + a_3 x_3)^{(3)}$.

2. The polar conic of P with regard to the curve k^3, represented by (1) has as equation

$$3 a^2_x a_y b^3_y + \lambda (2 a_x a^2_y b_x b^2_y + a^2_x a_y b^3_y) = 0,$$

or, if we put

$$a^2_x a_y \equiv K \qquad \text{and} \qquad a_x a^2_y \equiv L,$$

we find

$$(3 + \lambda) b^3_y K + 2 \lambda L^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is evident from this that the polar conics of P with respect to the curves of the pencil determined by k^3 and h^3 touch each other in their points of intersection with the polar line p^1, therefore the polar line of P with respect to all the curves $k^3{}_\lambda$ of this pencil.

For the curve $k^3{}_\lambda$ passing through P ensues from this that it must have a node in P.

Evidently the equation of this curve is

$$a^3{}_x\, b^3{}_y - a^2{}_x\, a_y\, b_x\, b^2{}_y = 0, \quad \ldots \ldots \quad (5)$$

whilst its polar conic is indicated by

$$a^2{}_x\, a_y\, b^3{}_y - a_x\, a^2{}_y\, b_x\, b^2{}_y = 0,$$

or by

$$b^3{}_y\, K - L^2 = 0,$$

from which is evident that it is composed of the tangents through P to the polar conic P with respect to k^3.

For $\lambda = -3$ we find a k^3 with the polar conic $L^2 = 0$. So it possesses three inflectional tangents meeting in P.

3. The satellite conic of P with respect to k^3 (that is the conic through the points where k^3 is intersected by the tangents drawn out of P) has for equation [1])

$$4\, a^2{}_x\, a_y\, b^3{}_y - 3\, a_x\, a^2{}_y\, b_x\, b^2{}_y = 0\,, \quad \ldots \ldots \quad (6)$$

or

$$4\, b^3{}_y\, K - 3\, L^2 = 0\,. \quad \ldots \ldots \ldots \quad (7)$$

To determine the satellite conic for the curve $k^3{}_\lambda$ we put

$$l^3{}_x \equiv a^3{}_x\, b^3{}_y + \lambda\, a^2{}_x\, a_y\, b_x\, b^2{}_y.$$

Then we find

$$3\, l^2{}_x\, l_y = (\lambda + 3)\, a^2{}_x\, a_y\, b^3{}_y + 2\, \lambda\, a_x\, a^2{}_y\, b_x\, b^2{}_y\,;$$

$$6\, l_x\, l^2{}_y = 2\, (\lambda + 3)\, a_x\, a^2{}_y\, b^3{}_y + 2\, \lambda\, (a^3{}_y\, b_x\, b^2{}_y + a_x\, a^2{}_y\, b^3{}_y),$$

or

$$l_x\, l^2{}_y = (\lambda + 1)\, a_x\, a^2{}_y\, b^3{}_y\,;$$

$$l^3{}_y = (\lambda + 1)\, a^3{}_y\, b^3{}_y.$$

So according to (6) the equation of the satellite of $k^3{}_\lambda$ is

$$4[(\lambda+3)\, a^2{}_x a_y b^3{}_y + 2\lambda a_x a^2{}_y b_x b^2{}_y]\, (\lambda+1)\, c^3{}_y d^3{}_y - 9\, (\lambda+1)^2\, a_x a^2{}_y b^3{}_y c_x c^2{}_y d^3{}_y = 0,$$

or as a, b, c and d are equivalent symbols,

$$(4\,\lambda + 12)\, a^2{}_x\, a_y\, b^3{}_y - (\lambda + 9)\, a_x\, a^2{}_y\, b_x\, b^2{}_y = 0,$$

or

$$(4\,\lambda + 12)\, b^3{}_y\, K - (\lambda + 9)\, L^2 = 0. \quad \ldots \ldots \quad (8)$$

From this ensues that the satellite conics and the polar conics of P with respect to the curves $k^3{}_\lambda$ belong to the same pencil. If we represent this by the equation

[1]) The deduction of this equation is found in SALMON "Higher plane curves". A stereometrical treatment of the satellite curves is found in the above-mentioned dissertation of Dr. H. DE VRIES, p. 18, 19 etc.

$$b^3{}_y\, K + \mu\, L^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

then

$\mu = 2\,\lambda : (\lambda + 3)$ furnishes the polar conic,

$\mu' = -(\lambda + 9) : (4\,\lambda + 12)$ the satellite conic of k^3).

Between the parameters μ and μ' exists the bilinear relation

$$\mu - 4\,\mu' = 3.$$

So for $\mu = -1$ and $\mu = \infty$ we find two curves k^3) for which polar conic and satellite coincide.

In the first case we have $\lambda = -1$; so we have the curve k^3) possessing in P a node.

In the second case we find $\lambda = -3$, so a curve for which the polar conic is a double right line.

For $\lambda = -9$ the satellite is indicated by $K = 0$. We then have the harmonic curve for which the satellite coincides with the polar conic of k^3; this well-known property indeed, ensues immediately from the definition of h^3.

4. Let us now consider the system of the satellite conics of a given point P with respect to the cubic curves of any pencil

$$A + \lambda\, B = 0.$$

By means of a selfevident notation the just mentioned system is represented by the equation

$$4\,(A_0 + \lambda\, B_0)\,(K_a + \lambda\, K_b) - 3\,(L_a + \lambda\, L_b)^2 = 0.$$

So through each point of the plane pass two satellites; the index μ is here *two*.

The satellite consists of two right lines when P is situated on the HESSIAN. Now the HESSIANS of the pencil evidently form a system with index three; the number of pairs of lines δ is therefore *three*.

A double line is found only when P lies on the cubic curve; consequently for our system η is equal to 1.

Between the characteristic numbers of a system of conics exist the wellknown relations

$$2\,\mu = \nu + \eta \qquad \text{and} \qquad 2\,\nu = \mu + \delta.$$

We find from the first $\nu = 3$, μ being equal to 2 and η to 1. The second then gives $\delta = 4$. From this ensues that the just mentioned satellite formed of two coinciding right lines must at the same time be regarded as a pair of lines, thus as a figure in which the centres of the two pencils of tangents have coincided.

From the equation

$$9\,(K_a + \lambda\, K_b)\,(L_a + \lambda\, L_b) - (A_0 + \lambda\, B_0)\,(A + \lambda\, B) = 0$$

it is evident that the harmonic curves of P with respect to the curves of the cubic pencil also form a system with index *two*.

For k^3 passing through P the curve h^3 breaks up into the system of the polar conic and the polar line of P with respect to that curve which touch each other in P.

As k^3 and h^3 have in common the tangents out of P, being thus of the same class, the harmonic curve has only then a node when this is the case with the original curve.

5. If with respect to a given k^3 we determine on each right line through P the points B_1, B_2, B_3 in such a way that B_i is harmonically separated by A_i from A_j and A_k, we get as locus of the points B a curve of order *six*, h^6, with a threefold point in P. For, if B_1 coincides with P, then A_1 is one of the points of intersection of k^3 with the polar line of P and the reverse (see § 1).

As the points B correspond one by one to the points A, the curve h^6 is of the same genus as k^3, so it has still 6 double points or cusps. This last is excluded because in that case not a single tangent could be drawn from P to h^6, whilst it is clear that the tangents out of P to k^3 also touch h^6.

From the definition of h^6 follows immediately that this curve can meet the curve k^3 only in the points of contact R of the above mentioned six tangents: so in each point R they have three points in common. The right line PR having in R two points in common with k^3, but three points with h^6, R must be one of the six nodes of h^6 and PR one of the tangents in that node.

Chemistry. — "*Preparation of cyclohexanol.*" By Prof. A. F. HOLLEMAN.

The preparation of ketohexamethylene in somewhat large quantities is one of the most lengthy operations, whatever known process may be used.

Since, by means of the addition of hydrogen to benzene, by the process of SABATIER and SENDERENS, hexa-hydrobenzene has become a readily accessible substance, it was thought advisable to use this as a starting point for the preparation of the said ketone by first converting it into monochlorohexamethylene, converting this in the usual manner into the corresponding alcohol and then oxidising this to ketone by the process indicated by BAEYER. Mr. VAN DER LAAN has tried, in my laboratory to realise this.