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Mathematics. — *"On complexes of rays in relation to a rational skew curve."* By Prof. J. DE VRIES.

(Communicated in the meeting of April 24, 1903).

1. Supposing the tangents of a rational skew curve R^n of degree n to be arranged in groups of an involution I^p of degree p, let us consider the complex of rays formed by the common transversals of each pair of tangents belonging to a group. So this complex contains each linear congruence the directrices of which belong to a group of I^p. If these directrices coincide to a double ray a of I^p the congruence evidently degenerates into two systems of rays, viz. the sheaf of rays with the point of contact A of a as vertex and the field of rays in the corresponding osculating plane α.

To find the degree of the complex let us consider the involution I^p of the intersections of the tangents with an arbitrary plane φ. The surface of the tangents intersects φ according to a curve C^m of degree $m = 2(n-1)$ and the complex curve of φ envelopes the lines connecting the pairs PP' of I^p. This involution having $(m-1)(p-1)$ pairs in common with the involution forming the intersection with an arbitrary pencil of rays, *the complex is of degree* $(2n-3)(p-1)$.

2. We then consider the correspondence between two points Q, Q' of C^m situated on a right line PP'. As Q lies on the lines connecting any of $(m-2)(p-1)$ pairs, there are $(m-2)(p-1)(m-3)$ points Q'. The correspondence (Q, Q') has $(m-2)(m-3)(p-1)^2$ pairs in common with I^p, so the *complexcurve* has

$$\tfrac{1}{2}(m-2)(m-3)(p-1)^2 = (n-2)(2n-5)(p-1)^2$$

double tangents, the *complexcone* as many *double edges*.

Evidently these double rays form a *congruence* comprised in the complex, of which *order and class are equal to* $(n-2)(2n-5)(p-1)^2$.

The complexcurve also possesses a number of threefold tangents, each containing three points of I^p belonging to one and the same group. To find this number we make each point of intersection S of C^m with the right line PP' to correspond to each point P'' of the group indicated by P. To each point P'' belong $\tfrac{1}{2}(p-1)(p-2)$ pairs P, P', so $\tfrac{1}{2}(p-1)(p-2)(m-2)$ points S; each point S lies on $(m-2)(p-1)$ connecting lines PP', and therefore it is conjugate to $(m-2)(p-1)(p-2)$ points P''. Every time P'' coincides with S, three points P lie in a right line and each of those points is a coincidence of the correspondence (P'', S); so we find $\tfrac{1}{2}(m-2)(p-1)(p-2)$ *threefold tangents*. From this appears at the same time that the right lines of which each cuts three tangents of R^n belonging to a same group of I^p, form *a congruence of which order and class are equal to* $(n-2)\ (p-1)\ (p-2)$.

3. Let us consider more closely the group where a is a double element and a' one of the other elements. To the just-mentioned congruence evidently belongs the pencil of rays in the plane $(A, a') \equiv \alpha_1$, with vertex A and the pencil of rays in the osculating plane α with vertex $(a, a_1) \equiv A_1$. So the congruence contains at the least $4(p-1)(p-2)$ *pencils of rays*; each of the $2(p-1)$ *singular points* A is the vertex of $(p-2)$ pencils placed in different planes; each of the $2(p-1)$ *singular planes* α bears $(p-2)$ pencils with different vertices; on the other hand the $2(p-1)(p-2)$ *singular points* A_1 and the $2(p-1)(p-2)$ *singular planes* α_1 each bear a pencil.

The complex curve is as appears from the above of genus $\frac{1}{2}[(2n-3)(p-1)-1][(2n-3)(p-1)-2]-(n-2)(2n-5)(p-1)^2-3(n-2)(p-1)(p-2)$. For $p=3$ this becomes equal to zero which could be foreseen; for, to each point P of the curve C^m the connecting line $P'P''$ can be made to correspond, by which the tangents of the complexcurve coincide one by one with the points of a rational curve.

In a plane φ through a tangent a' the complexcurve degenerates, a pencil of rays the vertex of which lies on the tangent a separating itself from the whole.

In a plane α evidently $(p-2)$ pencils of rays separate themselves.

4. We shall consider more closely the simplest case, where the complex is determined by a quadratic involution of the tangents of a skew cubic; $n=3, p=2$.

If A and B are the points of contact of the tangents a and b forming the double rays of the involution, and if α and β are the corresponding osculating planes, we assume as planes of coordinates $x_1=0,\ x_2=0,\ x_3=0,\ x_4=0$ successively the osculating plane α, the tangent plane (a, B), the tangent plane (b, A), the osculating plane β. The curve R^3 is then represented by

$$x_1 : x_2 : x_3 : x_4 = t^3 : t^2 : t : 1,$$

and for its tangents we have the relation

$$p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23} = t^4 : 2t^3 : 3t^2 : 1 : -2t : t^2.$$

The points A and B being indicated by the parameters $t=0$ and $t=\infty$, the parameters t and t' of the points of contact of two conjugate tangents satisfy the relation $t+t'=0$.

The coordinates of a common transversal of the tangents (t) and $(-t)$ evidently satisfy the conditions

$$\left.\begin{aligned} p_{12}-2tp_{13}+t^2p_{14}+3t^2p_{23}+t^4p_{34}+2t^3p_{42}=0,\\ p_{12}+2tp_{13}+t^2p_{14}+3t^2p_{23}+t^4p_{34}-2t^3p_{42}=0, \end{aligned}\right\}$$

therefore also

$$p_{12}+t^2(p_{14}+3p_{23})+t^4p_{34}=0 \qquad \text{and} \qquad t^2p_{42}=p_{13}.$$

By eliminating t we find the equation of the indicated complex:

$$p_{12}\,p^2_{42}+p_{13}\,p_{42}\,(p_{14}+3p_{23})+p_{34}\,p^2_{13}=0.$$

To this *cubic complex* belongs the linear congruence $p_{13}=0$, $p_{42}=0$. Its directrices l and m are represented by $x_1=0$, $x_3=0$ and $x_4=0$, $x_2=0$; the former connects A with the point (α, b), the latter unites B and (β, a).

Each ray of the congruence rests on two pairs of tangents, the corresponding parameters are determined by the equation

$$p_{34}t^4+(p_{14}+3p_{23})\,t^2+p_{12}=0.$$

So the complexcone has a double edge, the complexcurve a double tangent.

5. This is also evident in the following way. With given values of y_1, y_2, y_3, y_4 the equation $p_{42}=\lambda p_{13}$ or $y_2x_4-y_4x_2=\lambda(y_3x_1-y_1x_3)$ represents a plane intersecting the complexcone twice according to $p_{42}=0$, $p_{13}=0$, and moreover according to a right line of the plane

$$\lambda^2(y_2x_1-y_1x_2)+\lambda[(y_4x_1-y_1x_4)+3(y_3x_2-y_2x_3)]+(y_4x_3-y_3x_4)=0.$$

So $p_{13}=0$, $p_{42}=0$ is a double edge.

If the plane $y_2x_4-y_4x_2=\lambda(y_3x_1-y_1x_3)$ is to touch the complexcone along the double edge, the three planes

$$y_3x_1-y_1x_3=0 \quad , \quad y_4x_2-y_2x_4=0,$$

$$(\lambda^2y_2+\lambda y_4)x_1+(3\lambda y_3-\lambda^2y_1)x_2+(y_4-3\lambda y_2)x_3-(\lambda y_1+y_3)x_4=0$$

must pass through *one* right line, so

$$\begin{aligned} \lambda^2y_2+\lambda y_4&=\varrho y_3 \quad , & 3\lambda y_3-\lambda^2y_1&=\sigma y_4,\\ y_4-3\lambda y_2&=-\varrho y_1 \quad , & \lambda y_1+y_3&=\sigma y_2. \end{aligned}$$

must be satisfied.

By eliminating ϱ or σ we find

$$\lambda^2y_1y_2+\lambda(y_1y_4-3y_2y_3)+y_3y_4=0.$$

The roots of this quadratic equation determine the tangent planes of the complexcone along the double edge, which becomes a cuspidal edge when

$$4y_1y_2y_3y_4=(y_1y_4-3y_2y_3)^2,$$

that is when

$$y_1y_4=y_2y_3 \qquad \text{or} \qquad y_1y_4=9y_2y_3.$$

So these quadratic skew surfaces of which the first evidently passes through R^3 contain the vertices of the *complexcones having a cuspidal edge.*

6. For the points P of the R^3 this cone of course degenerates into the plane connecting P with the tangent p' in the conjugate point P' and a quadratic cone touching that plane.

For points on the right lines l and m the complexcone must consist of a plane counted double and the single plane $x_1=0$ or $x_4=0$. For, each ray in α and β belongs to the complex, whilst all right lines resting on l and m are double rays of the complex. Indeed the substitution $y_1=0$, $y_3=0$ in the equation of the complex gives the relation $x_1(y_2x_4-y_4x_2)^2=0$..

For points on one of the tangents a and b the complexcone breaks up into the plane α or β and into a quadratic cone touching it.

For a point of the intersection of α and β we find a degeneration into three planes.

For the complexcurves analogous considerations hold good; e. g. the complexcurve degenerates into three pencils of rays when the plane passes through AB.

7. The complexcone degenerates into a plane and a quadratic cone if the vertex lies in α or in β or on the surface of the tangents of R^3. In the former case α or β belong to it; in the latter the plane through the vertex P and the conjugate tangent p'.

To investigate whether there are more points for which such a degeneration takes place, we suppose that the equation of the intersection of the complexcone with $x_4=0$, thus that

$$-y_1y_4x_2^3+y_1^2x_3^3-y_3y_4x_1^2x_2+y_3^2x_1^2x_3+(y_2y_4-3y_3^2)x_2^2x_1+$$
$$+3y_1y_3x_2^2x_3-2y_1y_3x_3^2x_1-3y_1y_2x_3^2x_2+(y_1y_4+3y_2y_3)x_1x_2x_3=0,$$

is deducible to the form

$$(b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+2b_{12}x_1x_2+2b_{13}x_1x_3+2b_{23}x_2x_3)(c_1x_1+c_2x_2+c_3x_3)=0.$$

Then the following conditions are to be satisfied:

$$b_{11}c_1=0, \qquad b_{22}c_2=-y_1y_4, \qquad b_{33}c_3=y_1^2,$$
$$b_{11}c_2+2b_{12}c_1=-y_3y_4, \qquad b_{11}c_3+2b_{13}c_1=y_3^2, \qquad b_{22}c_1+2b_{12}c_2=y_2y_4-3y_3^2,$$
$$b_{22}c_3+2b_{23}c_2=3y_1y_3, \qquad b_{33}c_1+2b_{13}c_3=-2y_1y_3, \qquad b_{33}c_2+2b_{23}c_3=-3y_1y_2,$$
$$2(b_{12}c_3+b_{23}c_1+b_{13}c_2)=y_1y_4+3y_2y_3.$$

Let us in the first place put $b_{11}=0$ and $c_1=y_3$, then $2b_{12}$ is equal to $-y_4$ and $2b_{13}$ equal to y_3. Further we find $b_{33}=-y_1$ and $c_3=-y_1$. After some deduction we get as only condition

$$y_1^2y_4^2 + 4y_1y_3^2 - 6y_1y_2y_3y_4 - 3y_2^2y_3^2 + 4y_2^3y_4 = 0,$$

or

$$(y_1y_4 - y_2y_3)^2 = 4(y_1y_3 - y_2^2)(y_2y_4 - y_3^2),$$

that is *the vertex of the complexcone belongs to the surface of tangents.*

If we put $c_1 = 0$, we then arrive after excluding $y_1 = 0$ and $y_4 = 0$ (for which the indicated degeneration always takes place) at the double condition

$$y_2y_4 = y_3^2 \text{ and } y_1y_4 = y_2y_3,$$

that is at the points of R^3.

8. Let us suppose that the tangents of R^3 are arranged in the triplets of a J^3. To determine the degree of the complex of the common transversals of the pairs of tangents we can also set about as follows. In an arbitrary pencil we consider the correspondence of two rays s and s', which are cut by two tangents belonging to J^3. To the coincidences of this correspondence (8, 8) belong the four rays resting on the double rays a, b, c, d of J^3; the others are united in pairs to six rays, each resting on two tangents of a triplet, *so the complex is of degree* 6.

To find the degree of the congruence of the right lines, each resting on the three tangents of a group, let us consider the rays they have in common with the analogous congruence belonging to a second J^3. If r_1, r_2 is one of the four common pairs of the two involutions, and r_3 and r_3' successively the tangent forming with r_1 and r_2 a group, the common transversals of r_1, r_2, r_3 and r_3' belong to the two congruences [1]). Evidently they can have no other rays in common than those eight, which are indicated by these; consequently the congruence is of order two.

The complexcone of an arbitrary point P has as appears from the above, *two threefold edges;* as it has to be rational, it has moreover *four double edges.*

If P lies on the surface of tangents of R^3, this cone degenerates into the system of planes which connect P with the two tangents conjugate to p and a biquadratic cone with threefold edge.

9. The quadratic scrolls determined by the triplets of tangents, evidently form a system of surfaces two of which pass through any point and two of which touch any plane. This system is thus represented in point- or tangential coordinates by an equation of the form

[1]) This consideration leads to no result if we consider a rational skew curve of higher order.

$$P + 2\lambda Q + \lambda^2 R = 0.$$

From this ensues that all the surfaces of this system have the eight common points (tangential planes) of $P=0$, $Q=0$, $R=0$ in common.

The degenerations of this system are four figures consisting each of two planes as locus of points and of two points as locus of tangential planes. One of those figures is formed by the planes α and $\alpha_1 \equiv (A\,a')$ and the points A and $A_1 \equiv (a\,a')$.

The eight common points $A_2, B_2, C_2, D_2, A_3, B_3, C_3, D_3$ and the eight common tangential planes $\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3$ of the scrolls are *singular* for the congruence (2,2). The remaining singular points and planes are evidently $A, B, C, D, A_1, B_1, C_1, D_1$ and $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \gamma_1, \delta_1$. These 16 points and 16 planes form the well known configuration of KUMMER.

We can choose the notation in such a way, that A_2 bears the planes $\beta, \gamma, \delta, \alpha_1$ and A_3 the planes $\beta_1, \gamma_1, \delta_1, \alpha$, etc. Let us bear in mind that three osculating planes of R^3 intersect each other in a point of the plane of their points of contact and let us further mark the symmetry of the figure, we can then easily deduce from the preceding, that

in α the points $A,\ A_1,\ A_3,\ B_2,\ C_2,\ D_2,$
" α_1 " " $A,\ A_1,\ A_2,\ B_3,\ C_3,\ D_3,$
" α_2 " " $A_1,\ A_2,\ A_3,\ B,\ C,\ D,$
" α_3 " " $A,\ A_2,\ A_3,\ B_1,\ C_1,\ D_1,$

are situated, whilst

A bears the planes $\alpha,\ \alpha_1,\ \alpha_3,\ \beta_2,\ \gamma_2,\ \delta_2,$
A_1 " " " $\alpha,\ \alpha_1,\ \alpha_2,\ \beta_3,\ \gamma_3,\ \delta_3,$
A_2 " " " $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta,\ \gamma,\ \delta,$
A_3 " " " $\alpha,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \gamma_1,\ \delta_1.$

It is clear that for each of these 16 points the complexcone is composed of a plane counted double and a cone of degree four.

Mathematics. — "*The singularities of the focal curve of a curve in space.*" By Dr. W. A. VERSLUYS. (Communicated by Prof. P. H. SCHOUTE.)

In paper N°. 5 of the "K. A. v. W." at Amsterdam, Vol. XIII, I have deduced some formulae expressing the singularities of the focal developable and of the focal curve in function of the singularities of a plane curve.

In like manner it is possible to deduce the following formulae which express the singularities of the focal developable and of the