Mathematics. Extract of a letter of Mr. V. WILLIOT, to the Academy.

In his splendid work entitled: "Théorie, propriétés, formules de transformation et méthodes d'évaluation des intégrales définies" Mr. BIERENS DE HAAN takes as basis to determine the general formulae (143, 144, 145, 146) of page 134 a definite discontinuous integral the value of which has been established farther on in the work (Partie III, Méthode 9, N°. 16) at page 333 as

$$\int_0^\infty sin\,(px)\,cos\,(qx)\,\frac{dx}{x} = \begin{cases} \frac{\pi}{2}\ for\ p > q \\ \frac{\pi}{4}\ for\ p = q \\ 0\ for\ p < q \end{cases} \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

the value with respect to the discontinuity $p = q$ being the mean of the extreme values.

But he gives this result on page 133 in the form

$$\int_0^\infty sin\ rx\ cos\ qx\,\frac{dx}{x} = \begin{cases} \frac{\pi}{2}\ for\ q \leqq r \\ 0\ for\ q > r \end{cases}$$

so that in the continuation of his deduction we find that the term corresponding to $q = r$ amounts to double the value of the real value and that the general formulae of page 134 are to be rectified in this way as well as the applications.

Particularly on page 639 formula 1900 we find

$$\int_0^\infty \frac{sin\ x}{1-2\,p\ cos\ x+p^2}\,\frac{cos\ ax}{x}\,dx = \frac{\pi}{2p}\sum_a^\infty p^n = \frac{\pi}{2}\,\frac{p^{a-1}}{1-p},$$

whilst the exact value of this integral is

$$\frac{\pi}{2}\left[\frac{p^{a-1}}{2} + p^a + p^{a+1} + p^{a+2} + \ldots\right] = \frac{\pi}{4}\,p^{a-1}\,\frac{1+p}{1-p}.$$

And really writing after multiplication by p:

$$\int_0^\infty \frac{p\ sin\ x}{1-2\,p\ cos\ x+p^2}\,\frac{cos\ ax}{x}\,dx = \frac{\pi}{4}\,p^a\,\frac{1+p}{1-p},$$

it is sufficient to develop the first factor of the function of which the integral is to be found

$$\frac{p\ sin\ x}{1-2\,p\ cos\ x+p^2} = \sum_{k=1}^\infty p^k\ sin\ kx$$

to refind by means of the integral (1) the development of the second term of the equation:

$$\frac{\pi}{4}\, p^a \frac{1+p}{1-p} = \frac{\pi}{2}\left[\frac{p^a}{2} + p^{a+1} + p^{a+2} + \ldots\right].$$

It was in looking for a way to place in a form of a definite integral the general term of the series of LAMBERT modified by CLAUSEN:

$$\frac{1+x^n}{1+x^n}\, x^{n^2} = \frac{4}{\pi}\int_0^\infty \frac{\sin\alpha \cos(n+1)\,\alpha}{1-2x^n\cos\alpha + x^{2n}}\,\frac{d\alpha}{\alpha}$$

that I found this error.

It is easily seen that the rectification has to be extended to the whole N°. 12 of the method 41 of which the above mentioned integral forms a part and to any other application of the general formulae of page 134.

This paper was given to Dr. J. C. KLUYVER, who made the following communication about it:

The remarks of Mr. WILLIOT are on the whole correct. In the "Exposé de la Théorie etc." of BIERENS DE HAAN we really find on page 639

$$\int_0^\infty \frac{\sin x}{1-2p\cos x + p^2} \cdot \frac{\cos ax}{x}\,dx = \frac{\pi}{2} \cdot \frac{p^{a-1}}{1-p}$$

and this is incorrect whether a is an entire number or not.

Mr. WILLIOT now gives as an answer

$$\frac{\pi}{4}\, p^{a-1}\, \frac{1+p}{1-p},$$

and that will do for a as an entire number.

In the meanwhile he might have observed that this result neither holds good for a (not an entire number) and that we find for any possible positive a:

$$p<1:\quad \frac{\pi}{4} \cdot \frac{p^{[a-\delta]} + p^{[a+\delta]}}{1-p}$$

$$p>1:\quad \frac{\pi}{4} \cdot \frac{p^{-[a-\delta]} + p^{-[a+\delta]}}{p\,(p-1)}$$

(Here $[a]$ means the greatest entire number smaller than a).

(October 27, 1903).