Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW)

Citation:

J.C. Kluyver, Series derived from the series [formula]., in:
KNAW, Proceedings, 6, 1903-1904, Amsterdam, 1904, pp. 305-312

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through the forming of small epithelium plugs, growing downwards from the surface.

And while the whole uterus is trying to regain its normal shape by means of contraction of all its muscles throughout, the mucous membrane must of course shrink considerably; this process follows new lines, different from those which I have so far met with in any of the puerperal uteri of mammalia, hitherto examined.

How this involution process progresses will be described in details by Dr. W. KURZ in an exhaustive work, freely enriched with illustrations, and in which due attention is paid to the works of reference written on the subject.

Mathematics. — "*Series derived from the series* $\Sigma \frac{\mu(m)}{m}$." By Prof. J. C. KLUYVER.

By $\mu(m)$ we denote an arithmetical function of the integer m, which equals 0 if m be divisible by a square, and otherwise equals $+1$ or -1, according as m is a product of an even or of an odd number of prime numbers.

The series

$$\sum_{m=1}^{m=\infty} \frac{\mu(m)}{m} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} \cdot\cdot\cdot\cdot\cdot + \frac{1}{26} - \frac{1}{29} - \frac{1}{30} \cdot\cdot\cdot\cdot$$

was considered by EULER, who concluded that it converged towards 0, a theorem only recently proved by von MANGOLDT (1897) and by LANDAU (1899).

In this paper it will be shewn that in innumerable ways we may select from EULER's series infinite groups of terms, each of these groups again constituting a convergent series.

In fact we may assume a linear congruence

$$x \equiv h \; . \; . \; . \; . \; . \; (mod.\ b)$$

and from EULER's series retain only those terms the denominators of which are solutions of the congruence.

From

$$T_{1,0} = \sum_{m=1}^{m=\infty} \frac{\mu(m)}{m}$$

we get thus the new series

$$T_{b,h} = \sum_{m=0}^{m=\infty} \frac{\mu(mb+h)}{mb+h}$$

and it will be found that this series has a definite sum, whatever may be the integers b and h ($h<b$).

Firstly consider the case $h=0$ and suppose b to be prime. As it is necessary to start with finite series we write, g being any positive number,

$$T^{g}_{b,0}=\sum_{m=1}^{mb<g}\frac{\mu(mb)}{mb}\quad,\quad T^{g}_{b,h}\sum_{m=0}^{mb+h<g}\frac{\mu(mb+h)}{mb+h}.$$

Then, since $\mu(mb)$ is either 0 or $-\mu(m)$ according as m is divisible by b or not, we have

$$T^{g}_{b,0}=-\frac{1}{b}\sum_{h=1}^{h=b-1}T^{\frac{g}{b}}_{b,h}=-\frac{1}{b}\left(T^{\frac{g}{b}}_{1,0}-T^{\frac{g}{b}}_{b,0}\right),$$

or

$$T^{g}_{b,0}=-\frac{1}{b}T^{\frac{g}{b}}_{1,0}+\frac{1}{b}T^{\frac{g}{b}}_{b,0}, \quad \ldots\ldots \quad (A)$$

and in the same way

$$T^{\frac{g}{b}}_{b,0}=-\frac{1}{b}T^{\frac{g}{b^2}}_{1,0}+\frac{1}{b}T^{\frac{g}{b^2}}_{b,0},$$

$$T^{\frac{g}{b^2}}_{b,0}=-\frac{1}{b}T^{\frac{g}{b^3}}_{1,0}+\frac{1}{b}T^{\frac{g}{b^3}}_{b,0},$$

$$\ldots\ldots\ldots\ldots$$

Supposing g to lie between b^{α} and $b^{\alpha+1}$ we infer from these equations

$$T^{g}_{b,0}=-\sum_{k=1}^{k=\alpha}\frac{1}{b^k}T^{\frac{g}{b^k}}_{1,0} \quad \ldots\ldots\ldots \quad (B)$$

Now it follows from the identity

$$1=\sum_{m=1}^{m<g}\mu(m)\left[\frac{g}{m}\right],$$

that

$$\sum_{m=1}^{m<g}\frac{\mu(m)}{m}$$

is always finite and less than 2; hence equation (B) gives

$$\left|T^{g}_{b,0}\right|<\frac{2}{b-1}$$

however large the number g may be, and taking this inequality into account it is seen from equation (A) that we have necessarily

$$T_{b,0} = \lim_{q=\infty} T^{q}_{b,0} = 0.$$

The proof given here is easily extended to the case in which b is not prime but a product of unequal prime factors. Square factors are excluded, for then $T_{b,0}$ is identically zero.

Let c be a prime number not dividing b, then instead of equation (A) we can establish the relation

$$T^{g}_{bc,0} = -\frac{1}{c} T^{\frac{q}{c}}_{b,0} + \frac{1}{c} T^{\frac{q}{c}}_{bc,0},$$

and it will appear that $T^{q}_{bc,0}$ tends to zero, if we can show that $\lim\limits_{q=\infty} T^{g}_{b,0} = 0$. Now the latter theorem is proved for any prime number b, hence it must remain true if repeatedly we multiply b by other prime factors c.

In order to investigate the series $T_{b,h}$ we consider the series of functions

$$\sum_{m=1}^{m=\infty} \frac{\mu(m) z^m}{1-z^m}.$$

Evidently it converges for $|z| < 1$, and expanding each term into a power series, we find

$$\sum_{m=1}^{m=\infty} \frac{\mu(m) z^m}{1-z^m} = \sum_{m=1}^{m=\infty} z^m \sum_{d/m} \mu(d) = z,$$

for the sum $\sum\limits_{d/m} \mu(d)$, in which the summation is extended over all divisors d of m, unity and m itself included, is zero except for $m = 1$.

Similarly we have by changing m into mb

$$\sum_{m=1}^{m=\infty} \frac{\mu(mb) z^{mb}}{1-z^{mb}} = \sum_{m=1}^{m=\infty} z^{mb} \sum_{d/m} \mu(bd).$$

Let b be the product of the prime factors $p_1, p_2, \ldots . p_k$, then $\sum\limits_{d/m} \mu(bd)$ is zero but for those integers m that are of the peculiar form $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, and in that case we have $\sum\limits_{d/m} \mu(bd) = \mu(b)$.

Hence we may write

$$\sum_{m=1}^{m=\infty} \frac{\mu(mb)\, z^{mb}}{1-z^{mb}} = \mu(b) \sum_{\alpha_1, \alpha_2, \ldots \alpha_k} z^{nb},$$

$$(n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k})$$

(a relation still holding for any b having square factors).

Integrating from the above equations we deduce

$$\sum_{m=1}^{m=\infty} \frac{\mu(m)}{m} \log (1-z^m) = - z,$$

$$\sum_{m=1}^{m=\infty} \frac{\mu(mb)}{mb} \log (1-z^{mb}) = -\mu(b) \sum_{\alpha_1, \alpha_2, \ldots \alpha_k} \frac{z^{nb}}{nb}$$

and subtracting

$$\sum_{h=1}^{h=b-1} \sum_{m=0}^{m=\infty} \frac{\mu(mb+h)}{mb+h} \log (1-z^{mb+h}) = - z + \mu(b) \sum_{\alpha_1, \alpha_2, \ldots \alpha_k} \frac{z^{nb}}{nb}.$$

Denoting by k a number less than b and prime to b I substitute

$$z = \varrho\, e^{2\pi i \frac{k}{b}}$$

and afterwards make z tend to unity.

The righthand side ultimately takes the value of

$$- e^{2\pi i \frac{k}{b}} + \frac{\mu(b)}{b} \sum_{\alpha_1, \alpha_2, \ldots \alpha_k} \frac{1}{n} = - e^{2\pi i \frac{k}{b}} + \frac{\mu(b)}{b\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \ldots \left(1-\frac{1}{p_k}\right)} =$$

$$= - e^{2\pi i \frac{k}{b}} + \frac{\mu(b)}{\varphi(b)}$$

where $\varphi(b)$ indicates the number of integers less than b and prime to b and the equation itself may be written

$$\sum_{h=1}^{h=b-1} T_{b,h} \times \log \left(1 - e^{2\pi i \frac{hk}{b}}\right) = - e^{2\pi i \frac{k}{b}} + \frac{\mu(b)}{\varphi(b)} \quad . \quad . \quad . \quad (C)$$

It implies k prime to b, but for other integers k less than b and not prime to b a similar equation can be established.

Suppose

$$\frac{k}{b}=\frac{k'}{b'},$$

where k' and b' are now prime to each other, then we have

$$\sum_{h'=1}^{h'=b'-1} T_{b',h'} \times log\left(1-e^{2\pi i\frac{h'k'}{b'}}\right)=-e^{2\pi i\frac{k'}{b'}}+\frac{\mu(b')}{\varphi(b')}.$$

But denoting by h all integers less than b that satisfy the congruence

$$h \equiv h' \ldots (mod.\ b')$$

we have evidently

$$\sum_{h} T_{b,h} = T_{b',h'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

and also

$$\sum_{h} T_{b,h} \times log\left(1-e^{2\pi i\frac{hk}{b}}\right)=T_{b',h'} \times log\left(1-e^{2\pi i\frac{h'k'}{b'}}\right);$$

hence in case k is not prime to b we are led to the equation

$$\sum_{h=1}^{h=b-1} T_{b,h} \times log\left(1-e^{2\pi i\frac{hk}{b}}\right)=-e^{2\pi i\frac{k}{b}}+\frac{\mu(b')}{\varphi(b')}, \quad . \quad . \quad (E)$$

if only we omit at the lefthand side the terms corresponding to those integers h that are multiples of b'. With this limitation the equation (E) applies to all values of k, for if k be prime to b, from it we get back the equation (C).

In this way we obtain by putting successively $k=1, 2, \ldots b-1$ a set of $b-1$ equations, from which we find in the shape of determinants finite values for the $b-1$ quantities $T_{b,h}$.

Actually we have got more equations than were wanted, for we may separate real and imaginary parts.

We put

$$x-[x]-\frac{1}{2}=P(x),$$

so that $P(x)$ stands for the fractional part of the number x minus $\frac{1}{2}$. Now we have generally

$$log\,(1-e^{2\pi i x})=\frac{1}{2}\,log\,4\,sin^2 \pi x + i\,\pi\,P(x),$$

hence instead of (E) we get the two equations

$$\sum_{h=1}^{h=b-1} T_{b,h} \times \log 4 \sin^2 \pi \frac{hk}{b} = -2 \cos 2\pi \frac{k}{b} + 2 \frac{\mu(b')}{\varphi(b')}, \quad . \quad (F)$$

and

$$\sum_{h=1}^{h=b-1} T_{b,h} \times P\left(\frac{hk}{b}\right) = -\frac{1}{\pi} \sin 2\pi \frac{k}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (G)$$

Again the equation (F) supposes that if we have

$$\frac{k}{b} = \frac{k'}{b'}$$

no multiples of b' should be substituted for b in the summation at the lefthand side. As for the equation (G) this limitation is superfluous since the discontinuous function $P(x)$ vanishes for integer values of x.

As the solutions of the equation (F) and (G) seem in general to present neither regularity nor symmetry, we will proceed to consider some particular cases.

In the case $b = 2$, we have at once

$$T_{2,0} = \quad -\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} \cdots = 0,$$

$$T_{2,1} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} \cdots = 0.$$

Putting $b = 3$ and substituting $k = 1$ in (G) we find

$$-\frac{1}{6} T_{3,1} + \frac{1}{6} T_{3,2} = -\frac{\sqrt{3}}{2\pi},$$

and since

$$T_{3,1} + T_{3,2} = 0,$$

we have

$$T_{3,1} = 1 - \frac{1}{7} + \frac{1}{10} - \frac{1}{13} - \frac{1}{19} \cdots = \quad \frac{3\sqrt{3}}{2\pi},$$

$$T_{3,2} = \quad -\frac{1}{2} - \frac{2}{5} - \frac{1}{11} + \frac{1}{14} \cdots = -\frac{3\sqrt{3}}{2\pi}.$$

In the case $b = 6$ we may apply (D). Thus we obtain relations

$$T_{6,1} + T_{6,5} = T_{2,1} = 0, \quad T_{6,3} = T_{3,0} = 0, \quad T_{6,1} + T_{6,4} = T_{3,1} = \quad \frac{3\sqrt{3}}{2\pi},$$

$$T_{6,2} + T_{6,4} = T_{2,0} = 0, \qquad T_{6,2} + T_{6,5} = T_{3,2} = -\frac{3\sqrt{3}}{2\pi}.$$

Joining to these the equation resulting by substituting $b = 6$, $k = 1$ in (G)

$$(T_{6,1}-T_{6,5})\,P\left(\frac{1}{6}\right)+(T_{6,2}-T_{6,4})\,P\left(\frac{2}{6}\right)=-\frac{1}{\pi}\sin\frac{\pi}{3},$$

there follows

$$T_{6,1}=1-\frac{1}{7}-\frac{1}{13}-\frac{1}{19}-\frac{1}{31}\ldots=-\frac{\sqrt{3}}{\pi},\quad T_{6,2}=-\frac{1}{2}+\frac{1}{14}+\frac{1}{26}+\frac{1}{38}\ldots=-\frac{\sqrt{3}}{2\pi},$$

$$T_{6,5}=\quad-\frac{1}{5}-\frac{1}{11}-\frac{1}{17}-\frac{1}{23}\ldots=-\frac{\sqrt{3}}{\pi},\quad T_{6,4}=+\frac{1}{10}+\frac{1}{22}+\frac{1}{34}+\frac{1}{46}\ldots=+\frac{\sqrt{3}}{2\pi}.$$

If we take $b=4$, we have by applying (D)

$$T_{4,2}=T_{2,0}=0,$$

$$T_{4,1}+T_{4,3}=T_{2,1}=0,$$

and by the substitution $b=4$, $k=1$ in (G)

$$T_{4,1}\times P\left(\frac{1}{4}\right)+T_{4,3}\times P\left(\frac{3}{4}\right)=-\frac{1}{\pi},$$

hence we find

$$T_{4,1}=1-\frac{1}{5}-\frac{1}{13}-\frac{1}{17}+\frac{1}{21}\cdots=\quad\frac{2}{\pi},$$

$$T_{4,3}=\quad-\frac{1}{3}-\frac{1}{7}-\frac{1}{11}+\frac{1}{15}\cdots=-\frac{2}{\pi}.$$

And by subtraction we obtain

$$\sum_{m=0}^{m=\infty}\left[\frac{\mu(4m+1)}{4m+1}-\frac{\mu(4m+3)}{4m+3}\right]=\frac{4}{\pi},$$

a result which may be compared with LEIBNITZ's theorem

$$\sum_{m=0}^{m=\infty}\left[\frac{1}{4m+1}-\frac{1}{4m+3}\right]=\frac{\pi}{4}.$$

Lastly the series $T_{5,h}$ can be evaluated if we substitute $b=5$, $k=1$ in (F) and $b=5$, $k=1$, $l=2$ in (G). Thus we obtain

$$(T_{5,1}+T_{5,4})\log 4\sin^2\frac{\pi}{5}+(T_{5,2}+T_{5,3})\log 4\sin^2\frac{2\pi}{5}=-2\cos\frac{2\pi}{5}-\frac{1}{2},$$

$$(T_{5,1}-T_{5,4})\,P\left(\frac{1}{5}\right)\quad+(T_{5,2}-T_{5,3})\,P\left(\frac{2}{5}\right)\quad=-\frac{1}{\pi}\sin^2\frac{2\pi}{5},$$

$$(T_{5,1}-T_{5,4})\,P\left(\frac{2}{5}\right)\quad+(T_{5,2}-T_{5,3})\,P\left(\frac{1}{5}\right)\quad=-\frac{1}{\pi}\sin^2\frac{4\pi}{5},$$

moreover we have

$$T_{5,1}+T_{5,2}+T_{5,3}+T_{5,4}=-T_{5,0}=0.$$

Solving the four equations, the result is as follows

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$$T_{5,1} = 1 + \frac{1}{6} - \frac{1}{11} + \frac{1}{21} + \frac{1}{26} + \ldots = \frac{1}{2\pi}(3 \sin 72^\circ + \sin 36^\circ) - \frac{1 + 4 \cos 72^\circ}{8 \log 2 \sin 18^\circ} = + 1.128,$$

$$T_{5,2} = -\frac{1}{2} - \frac{1}{7} - \frac{1}{17} + \frac{1}{22} - \frac{1}{37} \ldots = \frac{1}{2\pi}(\sin 72^\circ - 3 \sin 36^\circ) + \frac{1 + 4 \cos 72^\circ}{8 \log 2 \sin 18^\circ} = - 0,710$$

$$T_{5,3} = -\frac{1}{3} - \frac{1}{13} - \frac{1}{23} + \frac{1}{33} + \frac{1}{38} \ldots = \frac{1}{2\pi}(- \sin 72^\circ + 3 \sin 36^\circ) + \frac{1 + 4 \cos 72^\circ}{8 \log 2 \sin 18^\circ} = - 0.452,$$

$$T_{5,4} = + \frac{1}{14} - \frac{1}{19} - \frac{1}{29} + \frac{1}{34} + \frac{1}{39} \ldots = \frac{1}{2\pi}(- 3 \sin 72^\circ - \sin 36^\circ) - \frac{1 + 4 \cos 72^\circ}{8 \log 2 \sin 18^\circ} = + 0.034.$$

As a numerical test I have directly calculated $\overset{100}{T_{5,h}}$. The results were respectively: +1.123, —0.685, —0.449, +0.036.

From these few particular cases it will be evident that the equations (F) and (G) always permit to evaluate $T_{b,h}$ and the fact that such a series has in all cases a finite sum may with more or less justice be interpreted thus: Among the integers without quadratic factors less than a given large number g, that are solutions of a given congruence

$$x \equiv h \;.\;.\;.\;. \;(mod.\; b)$$

the integers made up by an odd number of prime factors are sensibly equal in number to the integers made up by an even number of prime factors.

Botany. — *"The Ascus-form of Aspergillus fumigatus Fresenius".* By Dr. G. Grijns. (Communicated by Prof. F. A. F. C. Went.)

While during the last course I was occupied with determining fungi in the botanical laboratory under the superintendence of Prof. Went, I noticed that in a pure culture of Aspergillus fumigatus, a couple of months old, sporefruits had formed; on inoculation from this culture these same bodies were always produced in the new cultures.

As nutrient substance I used Koning's malt-canesugar-agar-agar.

The ascus-form of Aspergillus fumigatus has not yet been described, for the statements of Behrens and Siebenmann are justly doubted by Wehmer, nor do they agree with my result.