Astronomy. — *"Investigation of the errors of the tables of the moon of* HANSEN—NEWCOMB *for the years* 1895—1902". By Dr. E. F. VAN DE SANDE BAKHUIJZEN.

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1. Introduction.

1. In the years 1901—1902 Mr. C. SANDERS has made a longitude determination on the West-coast of Africa by means of the moon. The investigation described in the following pages was undertaken in order to furnish him with accurate data for the moon's places.

Especially with regard to the systematic errors which affect all the observations of the moon's limbs, it is desirable to use for this purpose not only a few observations made in the neighbourhood of the days for which the places of the moon are required, but to make a more extensive investigation of the errors of the tables. There was still another reason for doing so. For when I first undertook the work, for which the observations at Greenwich had to form the basis, I could dispose only of those up to the year 1899, so that a direct determination of the required corrections was entirely impossible.

So at first I employed only the observations of the years 1895—1899, but later I was able to extend my investigation also over the 3 following years. For this I am indebted to the courtesy of Mr. CHRISTIE, who sent me a complete copy of the observations of the moon made at Greenwich during the years 1900—1902 and who thus enabled me to render my results much more reliable. In the same letter, however, Mr. CHRISTIE told me that a similar investigation for a similar purpose had been undertaken at Greenwich [1]) and at first this made me doubt whether in this circumstance it would not be better to stop my work. But as my calculations for the period 1895—1899 were rather far advanced, I ultimately resolved to continue them. I considered that perhaps in this case it might be useful when two independent investigations should confirm each other.

2. It is well-known that the motion of the moon offers many unsolved problems. Quite recently NEWCOMB in a paper read at the March-meeting of the English Royal Astronomical Society [2]), (when I had already begun my work), once more clearly pointed out the deficiencies of the theory which chiefly his investigations had brought to light. Let us shortly recapitulate those investigations.

[1]) Comp. Report of the Astronomer Royal.... read 1903 June 6, p. 9.

[2]) Monthl. Not. R. Astr. Soc. Vol. 63, p. 316.

In 1876 NEWCOMB published a comparison of the observations of the moon from 1862—1874 with the tables of HANSEN [1]) and showed the existence of slowly increasing errors in the tabular mean longitude. On the other hand, after having applied theoretical corrections to the coefficients of some of HANSEN's inequalities of short period, he found a hitherto unsuspected inequality in the true longitude of the form $a \sin (g + N)$, where g represents the mean anomaly and N an angle increasing by about 20° per annum. The long period errors were further investigated by NEWCOMB in his *Researches* [2]), which appeared in 1878. After an elaborate investigation of all the observations before 1750, he embodied the errors found in an empirical formula, which apparently satisfied all the available observations.

In the same year he published his "*Corrections to* HANSEN's *tables of the moon*", where tables were given for the application of the long period corrections according to the empirical formula alluded to above and for the correction of a term of the true longitude accidentally introduced into the tables with a wrong sign. For the time being he did not consider it advisable to apply other corrections. These "*Corrections*" have since been introduced into all the lunar ephemerides.

For the empirical term of long period no theoretical basis has been found until now. As for the term depending on $g + N$, NEISON's and HILL's investigations have shown that it may be the "*Jovian Evection*".

II. Investigation of the errors of longitude.

3. In my investigation I followed the same method as NEWCOMB in his paper of 1876, that is to say, instead of the errors of longitude and latitude I used those of right ascension and declination. Although in this way the calculations become somewhat more intricate, it offers the great advantage that the errors of observation, the systematic and the chance errors, in the two coordinates do not become intermixed.

Thus in investigating the errors of longitude, I started from the differences $\Delta\, \alpha$, which, in accordance with NEWCOMB I take in the sense: Computation—Observation.

4. In the first place I had to investigate the systematic errors in the observed transits of the two limbs, but, as it is well-known, the values found for them depend to a high degree on the value adopted for the parallactic inequality. This renders an independent determination of the two very difficult, as, for instance, it may be

[1]) S. NEWCOMB, Investigation of corrections to HANSEN's tables of the moon with tables for their application. Washington 1876.

[2]) S. NEWCOMB, Researches on the motion of the moon. Washington 1878.

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seen from an examinination of NEWCOMB's elaborate investigation, laid down in his *"Astronomical constants"* p. 148—151.

Therefore I thought it best to leave out an independent determination of the coefficient of the parallactic inequality. For in the first place the investigations of the last years have yielded a value of the solar parallax that must be pretty accurate and in the second place the direct determinations of the parallactic inequality that are entirely or partly free from the disadvantage mentioned, namely those of BATTERMANN [1]) from occultations observed at Berlin and those of FRANZ [2]) from transit observations of the crater MÖSTING *A* made at Königsberg, give results which agree satisfactorily with the most probable value of the solar parallax.

Whilst as this most probable value we may consider $\pi = 8''.796$, the investigations of BATTERMANN and FRANZ yield:

BATTERMANN	1884—85	$\pi = 8''.794$
"	1894—96	8. 775
FRANZ	1892	8. 770 [3])

On the other hand NEWCOMB derived from the transit observations of the limbs by eliminating as far as possible the systematic errors:

NEWCOMB 1862—94 $\pi = 8.''802$.

I have adopted $\pi = 8''.796$, hence as the correction of the value used ultimately by HANSEN in the *Tables de la Lune*: $\delta\pi = -0''.120$, whence as correction of the coefficient of the principal term of the parallactic inequality in the mean longitude:

$$\delta P = - 14.10 \times \delta\pi = + 1''.69$$
$$\text{and } P = - 124''.01$$

The correction of the value adopted by NEWCOMB becomes:

$$\delta P = + 0''.73.$$

5. I now proceeded as follows. The residuals $\Delta\alpha$ for each year were arranged according to the observed limb and the true age of the moon expressed in days. In this way each year yielded 25 groups of residuals and for each of them the mean value was derived.

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1) H. BATTERMANN, Beobachtungs-Ergebnisse der Sternwarte zu Berlin No. 5. Berlin 1891.

H. BATTERMANN, Beobachtungs-Ergebnisse der Sternwarte zu Berlin No. 11. Berlin 1902.

2) Astron. Nachr. Bd. 136. p. 354.

3) I had overlooked the discussion by FRANZ of the observations of MÖSTING A made at GÖTTINGEN 1891—93 *(Astron. Nachr.* Vol. 144 p. 177). The combined result from KÖNIGSBERG and GÖTTINGEN is: $\pi = 8''.805$. (*Added* 1903 *Dec.*)

These annual means for each day of the moon's age had then to be corrected 1st for the correction of the parallactic inequality, 2nd for the small theoretical corrections derived by NEWCOMB on p. 10 of his "*Investigation*". However, for these annual means some of the latter might be neglected and besides I might identify annual means of $\Delta \alpha$ with those of the residuals in mean longitude and also annual means of the true age with those of the mean age, the argument D. [1])

For the years 1895—97 I had at my disposal not only the results of the transit observations, but also those obtained with the altazimuth. As however in this case the advantage of also using observations made at small elongations is more than balanced by the difficulty of determining their systematic errors, I have ultimately only used the transit observations [2]).

From these corrected means, which are not given here, I further derived mean values for each year for each of the two limbs. They were obtained by combining with equal weights the results for each day of the age of the moon. However, for reasons to be given hereafter, the values for the ages of 4 and 26 days were ultimately rejected altogether.

In this way I found:

	$\Delta \alpha$ I	$\Delta \alpha$ II	II — I	$\frac{\text{I} + \text{II}}{2}$
1895	— 0s062	— 0s072	— 0s010	— 0s067
1896	— 0 131	— 0.044	+ 0 087	— 0.088
1897	— 0.134	— 0 126	+ 0.008	— 0 130
1898	— 0.177	— 0 104	+ 0 073	— 0.140
1899	— 0.125	— 0.070	+ 0.055	— 0.098
1900	— 0 151	— 0.104	+ 0.047	— 0.128
1901	— 0.144	— 0.091	+ 0.053	— 0.118
1902	— 0.189	— 0.123	+ 0.066	— 0.156
Mean			+ 0s047	

1) The corrections actually applied were the annual means for each value of D of the values of NEWCOMB's Table VII, after they had been corrected for the adopted value of the principal term of the parallactic inequality.

2) At first I had also used the altazimuth observations and from results obtained on the same day with both instruments I had derived for $\Delta\alpha_A - \Delta\alpha_T$ for observations of the 1st limb + 0s.126, and for those of the 2nd limb —0s.122.

By subtracting these annual means for each limb as they are given in the second and third columns of the table above from the means for the same limb for each day of the moon's age I obtained for each year a set of about 25 residuals and finally I combined the corresponding residuals of the 8 years with their respective weights.

These mean residuals follow here.

Limb I			Limb II		
Age	$\delta \Delta \alpha$	Weight	Age	$\delta \Delta \alpha$	Weight
4	— 0^s121	12	14	— 0 027	14
5	— 069	32	15	— 065	33
6	— 022	36	16	+ 023	61
7	— 036	41	17	+ 030	49
8	+ 022	50	18	— 009	53
9	+ 007	46	19	+ 048	45
10	+ 002	43	20	+ 008	34
11	+ 011	50	21	+ 003	31
12	+ 011	54	22	+ 013	37
13	+ 029	49	23	+ 017	38
14	+ 007	49	24	— 015	31
15	.000	27	25	+ .008	19
			26	+ .126	8

If we assume the adopted values for the inequalities depending on *D* to be correct, the two preceding tables show us the effect of the systematic errors of the observations. At a first glance at the second table we perceive that the right ascensions observed at the age of 4 and 26 days show abnormally great discordances, which both agree in sign with those which would result if the observers estimated the moon's diameter to be smaller when observed at daylight.

If we except these two groups, the observations of the 2nd limb no longer show any regular increase, whilst the results for the 1st limb between the ages of 5 and 8 days still seem to vary somewhat regularly. However, after due consideration of the case, I have ultimately assumed the personal error to be constant for the first limb between the age of 5 and 15 days and for the second limb between those of 14 and 25 days and I simply rejected the observations at the age of 4 and 26 days which are few in number. Perhaps it would have been better to apply a special correction at least to the results at the age of 5 days.

As stated above, the values in the 2[nd] and 3[d] columns of the first table are those found after the rejection of the two extreme groups, and from them I derived further the differences II—I and the values of $^1/_2$ (I + II). The II—I represent the differences between the personal errors for the two limbs. In the first three years these differences show considerable variations, but for the last five years there is a good agreement. However, as at first I only discussed the period 1895—1899 I adopted a mean value of II—I for these years and another for the following three, and for the corrections to be applied to the observations of the first and the second limbs to reduce them to the mean of the two I assumed for 1895—1899 $\pm$ $0^s.02$, for 1900—1903 $\pm$ $0^s.03$.

For a closer investigation of the personal errors it would be necessary to discuss separately the results of the different observers.

6. After having applied the corrections for personal error we must now compute for the separate observations the corrections to be applied to the mean longitude. in the first place those resulting from the corrections of the parallactic inequality of the annual equation, of the variation and of the evection — the last three as derived by Newcomb — and secondly the long period corrections. From the corrections of the mean longitude we must then derive those of the right ascensions.

The corrections of the first kind (comp. Newcomb *Invest.* p. 10 and 37 and Battermann N°. 5 p. 21) are, using Hansen's notations:

$$n\,\delta z = +\,1.''69 \sin D + 0.''16 \sin (D - g) - 0.''24 \sin (D + g')$$
$$+\,0.''09 \sin g' - 0.''33 \sin 2D - 0.''21 \sin (2D - g).$$

For the application of these corrections I have calculated 2 tables, partly arranged as Newcomb's Table VII and VIII.

For the long period corrections I first tried to derive accurate values from the whole available material.

For although the empirical correction derived by Newcomb in his *Researches,* has reduced the differences from the observations to a much smaller amount, there still remain unaccounted for discrepancies. This has been shown by Tisserand in his very lucid account of the questions involved here "Sur l'état actuel de la theorie de la lune" in the 3[d] volume of his Mécanique Céleste. He also showed there that we cannot improve the agreement by altering the period of the empirical term which NEWCOMB fixed on 273 years.

Hoping to find some indication about the empirical law which would represent the outstanding differences, I put together for the whole period 1847—1902 the values of the mean annual errors in longitude or in R. A. according to the observations at Greenwich. Those as far as 1882 [1]) were borrowed from STONE's papers in the *Monthl. Not.*, applying to his results NEWCOMB's corrections, while those for the subsequent years were taken from the Annual Reports of the Astronomical Society. To all these results the small corrections were applied for the reduction of the observations to the same standard time-star catalogue. As such I adopted the 2nd 10 year Catalogue.

I added to the Greenwich results: for the years 1862—1874 NEWCOMB's results which partly depend on the Washington observations, for the years 1880—1892 the results of the observations at Oxford as given by STONE, applying to both NEWCOMB's corrections and for the years 1895—1902 the results derived from the Greenwich observations by myself ($\frac{1}{2}$ (I + II) in the first table of section 5). From a comparison of the results of different observatories for the same year we may infer that they are tolerably accurate. The differences between my results and those computed at Greenwich range from 0."00 to 0."36.

I do not, however, give these annual mean errors here, as I did not succeed in deriving anything from them with certainty. By assuming the existence of a new inequality with a period of about 50 years with maxima about 1862 and 1887 and a coefficient of about 3" we should attain a somewhat better, but even then not an absolute agreement.

So the only thing I could do to obtain the mean corrections required for my purpose, was to represent the annual mean errors from 1886—1902 by a smooth curve. The following values were derived from it.

1895.0	$\delta\lambda = +$ 0"53
1896.0	1.06
1897.0	1.44
1898.0	1.72
1899.0	1.93
1900.0	2.09
1901.0	2.21
1902.0	2.28
1903.0	2.30

[1]) For the years 1847—1861 the new reduction of the Greenwich observations of the moon (*Monthl. Not.* Vol. 50) was used.

Obviously the last values cannot be very certain.

After thus having formed the total corrections to be applied to the mean longitude, they have been reduced to corrections of the right ascensions. For this reduction I could use the values F and $(v. \alpha)$ given by NEWCOMB in his Table IX and XI. The very small reduction from orbit longitude to ecliptic longitude could be neglected. (Comp. also *Investigation* p. 12 and 14).

7. The $\Delta\ \alpha$ corrected in this way were now used to derive from them the corrections of the true longitude, which depend on the sine and cosine of the mean anomaly. In his *Investigation* p. 16 NEWCOMB has shown that for this purpose we may use instead of the residuals of true longitude those of right ascension and although the error of the longitude of the node which is assumed to be small has increased since 1868, his conclusion still holds.

For each year the $\Delta\ \alpha$ were arranged in 18 groups according to the values of the mean anomaly, the first group containing those between $g = 0°$ and $20°$, the second those between $g = 20°$ and $40°$ etc. Then the sums and the means for each group were formed and were regarded as corresponding to $g = 10°$, $g = 30°$ etc. just as had been done by NEWCOMB.

If we represent the corrections which are to be applied to the true longitude of HANSEN by

$$\delta l = -h \sin g - k \cos g$$

we obtain for each year 18 equations of the form

$$c + h \sin g + k \cos g = r$$

where c is the outstanding mean error of longitude, whilst for h and k the signs are in accordance with NEWCOMB.

The equations were solved for each year by least squares with due regard to the weights of r, which were assumed to be proportional to the number of observations used.

So I obtained the following values of h and k:

	h	k
1895.5	+ 0''29	+ 0''44
1896.5	+ 0.66	+ 1.16
1897.5	+ 0.57	+ 1.77
1898.5	+ 0.51	+ 2.10
1899.5	— 0.93	+ 2.83
1900.5	— 1.66	+ 1.12
1901.5	— 1.46	+ 0.52
1902.5	— 1.18	+ 0.01

It is obvious that these coefficients cannot result from errors in the excentricity and the longitude of the perigee only, and their periodic character fully confirms the existence of the inequality discovered by NEWCOMB.

At a closer inspection, however, it appears that NEWCOMB's formula does not represent satisfactorily my h and k, and this need not astonish us if we consider the great extrapolation involved in the application of NEWCOMB's formula to my results.

8. To correct NEWCOMB's formula by successive approximations I have proceeded in the following way

By comparing the h and k now obtained with those in the table in *Investigation* p. 28, it may be easily seen that the period of the argument N, on which h and k depend through the formulae $h = h_c - a \sin N$ and $k = k_c + a \cos N$, must be greater than $16^2/_3$ years — the period assumed by NEWCOMB — and cannot differ much from 18 years. This corresponds to an annual variation of 20° and it will be convenient to adopt this value as a first approximation.

The special aim of my first operation was to find reliable values for the constant parts of the coefficients, h_c and k_c I tried to attain this by calculating values of h and k for each year of the 18 year-cycle by means of the results of NEWCOMB's two series and of those found for 1895—1902.

Assuming the argument for 1862 0 $\pm n \times 18$ to be 0, I derived normal values for the arguments 0 5, 1 5 etc. to 17 5, assigning the weights 1, 3 and 2 to the results of the 3 series I had no value for the argument 14 5 and therefore had to form it by interpolation.

In this way I found

Arg	h	k	Arg	h	k
0 5	+ 0″23	+ 1″58	9 5	+ 1″51	— 0″74
1 5	— 0 76	+ 2 20	10 5	+ 1 97	— 0 45
2 5	— 1 31	+ 1 10	11 5	+ 1 67	+ 0 09
3 5	— 1 20	+ 0 12	12 5	+ 1 79	+ 0 77
4 5	— 0 69	— 0 06	13 5	+ 0 80	+ 1 24
5 5	— 0 79	— 0 68	14 5	+ 0 80	+ 0 84
6 5	+ 0 20	— 1 57	15 5	+ 0 29	+ 0 44
7 5	+ 1 21	— 1 68	16 5	+ 0 66	+ 1 16
8 5	+ 1 20	— 1 46	17 5	+ 0 57	+ 1 77

From these values formulae were derived, which after a transformation in order to bring the zero-epoch on 1868.5 become

$$h = +0''.45 - 1''.30\ sin\ [167^\circ.1 + 20^\circ\ (t - 1868.5)]$$
$$k = +0''.26 + 1''.46\ cos\ [149^\circ.3 + 20^\circ\ (t - 1868.5)]$$

If we assume that the amplitude and the argument of the two periodic terms must be equal, the formulae become

$$h = +0''.45 - 1''.37\ sin\ [157^\circ.7 + 20^\circ\ (t - 1868.5)]$$
$$k = +0''.26 + 1''.37\ cos\ [157^\circ.7 + 20^\circ\ (t - 1868.5)].$$

The object of the second operation was to derive from the observations 1895—1902 the most reliable value of N for the middle-epoch, assuming its annual variation to be 20°. Starting from the 8×18 values of r and assuming as known only the values of c, (as found from the solution of the equations for each year) and those of h_c and k_c (as found above), I first subtracted the c from the r and then freed the latter from the influence of h_c and k_c.

The residuals must then be of the form:

$$r' = -a\ sin\ N\ sin\ g + a\ cos\ N\ cos\ g = a\ cos\ (g + N_0 + t \times 20^\circ)$$

and now it is clear that the 8×18 residuals correspond with only 18 different values of the argument $N_0 + g + t \times 20^\circ$. Consequently these residuals could be combined in 18 values, for instance the r' for $g = 10^\circ$ in 1895 could be combined with that for $g = 350^\circ$ in 1896, with that for $g = 330^\circ$ in 1897 etc.

Having due regard to the weights, the following mean values of r' were derived. The arguments g hold for 1898 i.e. for 1898.5.

g	r'	g	r'	g	r'
10°	+ 1''61	130°	— 1''32	250°	+ 0''13
30	+ 1.68	150	— 1.29	270	+ 0.04
50	+ 0.65	170	— 1.31	290	+ 0.62
70	+ 0.23	190	— 1.05	310	+ 1.27
90	— 1.12	210	— 1.09	330	+ 1.66
110	— 1.17	230	— 0.77	350	+ 1.48

These values are represented by the formula.

$$-0''.42\ sin\ g + 1''.51\ cos\ g = +1''.57\ cos\ (g + 15^\circ.5).$$

and 15°.5 will be a pretty accurate normal value of N for 1898.5. For the derivation of a similar normal value from each of the two series of NEWCOMB I chose a less direct but simpler method. In each series I reduced the N derived from each year to a mean epoch by means of the annual variation 20° and then combined them with the weights as given by NEWCOMB [1]). I did not however use the N of

[1]) Applying the same method to the observations 1895—1902 I should have found for N_0 16°.9 instead of 15°.5.

NEWCOMB, but the slightly modified values, which were obtained by taking $h_c = + 0''.45$ and $k_c = + 0''.26$.

The three normal values obtained thus were:

1852.6	$N =$ 200 7	Weight 1	*O. — C.*	— 9°0
1868.5	161.9 [1])	3		+ 4.6
1898.5	15.5	2		— 2.3

and from them I derived a corrected formula for N; I found:

$$N = 157°.3 + 19°.35\,(t - 1868.5)$$

or taking the mean year as zero-epoch

$$N = 302°.4 + 19°.35\,(t - 1876.0).$$

The outstanding differences Obs.—Comp. are given above.

If I had assigned equal weights to the three normal values, I should have found for the annual motion 19°.45, while by excluding the first I should have found 19°.12, both differing only slightly from the most probable value.

At first when NEWCOMB's value for the annual variation of N appeared to be too large I had thought that the true value might be equal to the theoretical annual variation of the argument of the Jovian Evection, i. e. 20°.65. It appears, however, that even the latter is too large to satisfy the observations.

To judge in how far this is the case a comparison is given below of the values of N for each year as directly derived from observations, first with my formula, secondly with the formula we obtain if we assume the same value of N for 1876.0, but take as annual variation 20°.65. The two sets of differences are given under the headings $N_O - N_C$ and $N_O - N_J$.

Epoch	*Weight*	$N_O - N_C$	$N_O - N_J$
1847.8	1	— 56°	— 19°
48.9	3	+ 11	+ 46
50.1	3	+ 2	+ 36
51.2	3	— 22	+ 10
52.4	4	— 30	+ 1
53.5	3	— 30	— 1
54.6	3	— 37	— 9
55.8	0.5	+ 7	+ 33
56.9	3	+ 27	+ 52
58.1	1	+ 101	+ 124

[1]) With NEWCOMB's values of N we should have found 200.°5 and 161.°7.

Epoch	*Weight*	$N_O - N_C$	$N_O - N_J$
1862.5	3	— 18	0
63.5	5	— 25	— 9
64.5	5	— 19	— 4
65.5	4	+ 7	+ 21
66.5	2	— 19	— 7
67.5	4	— 24	— 13
68.5	4	+ 19	+ 29
69.5	5	+ 37	+ 45
70.5	5	+ 20	+ 27
71.5	3	+ 28	+ 34
72.5	4	+ 22	+ 27
73.5	4	+ 12	+ 15
74.5	4	+ 10	+ 12
1895.5	0.5	+ 82	+ 57
96.5	2	+ 8	— 19
97.5	4	— 3	— 31
98.5	4	— 20	— 49
99.5	6	— 9	— 40
1900.5	4	+ 12	— 20
01.5	4	+ 6	— 27
02.5	4	+ 4	— 30

That the differences, even those with the formula that is made to represent the observations as well as possible, are not altogether accidental, may be seen from the great number of permanencies of sign. Yet I hold that we are entitled to the conclusion that an annual variation of N of 19°.35 better represents reality than one of 20°.65.

Having thus derived a formula for N representing as well as possible the results at my disposal, I had still to correct the adopted values of the coefficient a and of h_c and k_c.

To this end I compared the observed values of h and k with the formulae

$$h = +0''.45 - 1''.50 \sin [302°.4 + 19°.35 (t - 1876.0)]$$
$$k = +0''.26 + 1''.50 \cos [302°.4 + 19°.35 (t - 1876.0)]$$

and formed the outstanding residuals Obs.—Comp. These residuals which for shortness are not given here, were divided into 4 groups according to the 4 quadrants of N, and for each of these groups mean values were formed which follow here:

	δh	δk
I	— 0″08	+ 0″41
II	+ 0.26	— 0.02
III	— 0.14	— 0.35
IV	— 0.62	— 0.13

Hence:

$$\delta h_c = -0''14$$
$$\delta k_c = -0''02$$

$$\delta a = -0''36 \text{ according to the } h$$
$$= +0.25 \quad ,, \quad ,, \quad ,, \quad k$$
$$\text{Mean value of } \delta a = -0.05$$

The two values for a obtained in this way do not agree satisfactorily. The mean value $a = 1''.45$, however, differs little from that deduced above by assuming the annual variation of N to be 20°.

The values of h_c and k_c remain also more or less uncertain. The δh and δk show a systematic character even to a higher degree than the δN, but I did not succeed in finding the real law of the discordances. If, for instance, we assume that h_c and k_c vary proportionally with the time, the agreement does not improve.

As the most probable results of my investigation I adopt:

$$h = +0''.31 - 1''.45 \sin [302°.4 + 19°.35 (t - 1876.0)]$$
$$k = +0''.24 + 1''.45 \cos [302°.4 + 19°.35 (t - 1876.0)]$$

Thence follow as corrections of the eccentricity and of the longitude of the perigee:

$$\delta e = -0''.16$$
$$e \delta \pi = +0''.12$$
$$\delta \pi = +2''.2$$

while an eventual correction of the motion of the perigee remains entirely uncertain.

The correction of the true longitude of the moon thus becomes:

$$\delta l = -0''.31 \sin g - 0''.24 \cos g +$$
$$+ 1''.45 \sin [g + 212°.4 + 19°.35 (t - 1876.0)].$$

With this formula we may compare the two results, which BATTERMANN derived from his occultations and which hold for about 1885.0 and 1896.0.

BATTERMANN found for the total corrections depending on g (comp. n°. 5 p. 41, n°. 11 p. 52).

$$1885.0 \quad \delta l = -1''.14 \sin g + 2''.67 \cos g$$
$$1896.0 \quad ,, = -0''.90 \sin g - 1''.35 \cos g$$

while from my formula would follow:

$$1885.0 \quad \delta l = +0''.99 \sin g + 0''.41 \cos g$$
$$1896.0 \quad \text{,,} \; = -1''.05 \sin g - 1''.49 \cos g$$

Thus we find a very good agreement for 1896.0, but the results for 1885.0 cannot at all be brought to harmonize. It would be very interesting to investigate also the meridian observations of the years about 1885.

The annual variation found for N agrees in absolute value almost exactly with that of the longitude of the node and we might put for the argument of the inequality: $g - \theta + 216°$. It is probable, however, that we have only to do here with a casual agreement.

The theoretical value of the "Jovian Evection" is according to the most accurate calculation by HILL:

$$\delta l = +0''.90 \sin [g + 238° + 20°.65 (t - 1876.0)].$$

For 1856 the theoretical argument and that of the empirical term are in good agreement, but in the following years they are more and more discordant.

9. It only remains now to put together the final results for the mean corrections of the longitude, as they were derived from the solution of the equations for each year.

In the following table the column headed $\delta\lambda$ contains the residual corrections found after the corrections derived from my curve had been applied, while the column headed $\delta\lambda_N$ contains the total corrections to be applied to the longitude of HANSEN-NEWCOMB.

	$\delta\lambda$	$\delta\lambda_N$
1895.5	— 0''07	+ 0''73
96.5	— 0.05	+ 1.21
97.5	+ 0.36	+ 1.95
98.5	+ 0.22	+ 2.05
99.5	— 0.48	+ 1.53
1900.5	— 0.27	+ 1.88
01.5	— 0.46	+ 1.79
02.5	+ 0.16	+ 2.46

The mean value of the $\delta\lambda$ amounts to $-0''.07$, which might be applied as a constant correction to the results according to my curve.

These results of the transit observations, which contain the unknown personal errors in observing the moon's limbs may be compared with those of BATTERMANN and also with those derived by FRANZ from the observations of the crater Mösting A. We then find that the results found by them for the mean longitude for 1885.0, 1896.0 and

1892.5 after being reduced on the system of the 2nd 10 year catalogue are greater by + 0."6, + 0."1 and + 0."5 respectively than those of the transit observations at Greenwich [1]).

As to the occultations, it is proved by H. G. v. d. SANDE BAKHUYZEN [2]) that values for the moon's longitude derived from them will generally be too small and therefore it is probable that the moon's longitude according to the observations at Greenwich is still in need of a positive correction.

III. *Investigation of the errors of the latitude.*

10. My investigation of the errors of the moon's latitude was based on that of the errors in declination.

First I tried to determine the constant errors in the observations of the moon's declination and to this end I utilized the observations from 1895 to 1899. From the differences $\Delta\delta$ = Comp.—Obs. I derived mean values for each of the two limbs for each month of the year and from them annual means were derived by taking the mean of the monthly means without regard to their weights.

In this way I obtained the results given in the following table. The 2nd and 3rd columns contain the annual means for the north limb and the south limb, the 4th the means of the two, the 5th their differences i. e. the errors of the moon's diameter, while the 6th contains this same error derived only from simultaneous observations of the two limbs near full moon.

	North	*South*	$\frac{N+S}{2}$	$N-S$	$(N-S)_f$
1895	— 0"15	+ 0"55	+ 0"20	— 0"70	— 0"65
1896	— 0.15	— 0 49	— 0 32	+ 0 34	+ 0 19
1897	— 0.55	+ 0 29	— 0.13	— 0 84	— 1.57
1898	+ 0 05	— 0.03	+ 0.01	+ 0.08	+ 0.78
1899	+ 0.35	+ 0.08	+ 0.22	+ 0.27	+ 0.16
Mean	— 0"09	+ 0"08	0"00	— 0"17	— 0"22

1) If we combine FRANZ's result from his Königsberg observations with that which he derived from those at Göttingen, which had been overlooked by me, the last difference, instead of + 0."5, becomes + 0."3 (*Added* 1903 Dec.)

2) H. G. v. d. SANDE BAKHUYZEN: The relation between the brightness of a luminous point and the moments at which we observe its sudden appearance or disappearance. Proc. Acad. Amst. 4. 465.

Although the differences between the results of the separate years seem to be real, I have applied only to the $\Delta \delta$ derived from observations of the north and the south limb the constant corrections $+0''.1$ and $-0''.1$.

For the observations of 1900—1902 I did not know which limb was observed. While, however, in the preceding years the constant errors appeared to be small and in the mean for the two limbs were found to be $0''.0$, I thought myself justified in neglecting them altogether for 1900—1902.

11. In the second place the $\Delta \delta$ had to be corrected for the errors of longitude.

We find to a sufficient degree of approximation (comp. also *Investigation* p. 31—32 [1]) that the derivative of the declination relatively to the mean longitude is:

$$\frac{d\delta}{d\lambda} = a\,(1 + c - c \cos 2\lambda) \cos \lambda + b \cos (\lambda - \theta)$$
$$+ 2\,a\,e \cos (2\lambda - \pi) + 2\,b\,e \cos (2\lambda - \pi - \theta)$$

where

$$a = \sin \varepsilon \quad = 0.398$$
$$b = \cos \varepsilon \sin i \quad = 0.083$$
$$c = \tfrac{1}{4} \sin^2 \varepsilon \quad = 0.040$$

For our purpose we may neglect the 3d and the 4th terms; their short periods permit of their influence being regarded as fortuitous. Also the 2d term has provisionally been neglected, as its influence [2]), may easily be accounted for afterwards.

So there only remains the 1st term, which has been tabulated by Newcomb in his Table XI, and I multiplied it by the total errors of the mean longitude. The errors of the true longitude depending on g, give rise in $d\delta$ only to terms of very short and of very long period which could be neglected as being without influence on the results to be derived.

12. The $\Delta \delta$ corrected in this way were arranged for each year into 18 groups according to the values of the argument of the latitude u, in the same way as it was done for the $\Delta \alpha$ according to the values of g, and then the sums and the means for each group were formed.

I do not give here these annual means, but only the general means derived from the total sums.

[1]) In the formula on p. 32 $3\,e\,K$ and $3\,e\,H$ ought to be $2\,e\,K$ and $2\,e\,H$.

[2]) This term is the influence of the error in longitude on the latitude and consequently directly influences the determination of the longitude of the node, but not that of the inclination.

u	$\Delta\,\delta$	u	$\Delta\,\delta$
10°	+ 0″67	190°	— 0″93
30	+ 0.70	210	— 0.57
50	+ 1.11	230	— 0.42
70	+ 0.75	250	— 0.06
90	+ 0.81	270	0.00
110	— 0.11	290	+ 0.39
130	— 0.01	310	+ 0.58
150	— 0.54	330	+ 1.05
170	— 1.11	350	+ 0.81

Each of the mean residuals gives an equation of condition:

$$\Delta\,\delta = -\,0.96\,\sin(\lambda-\theta)\,\delta i + 0.96\,\cos(\lambda-\theta)\,i\,\delta\theta$$

where δi and $\delta\theta$ represent the corrections to the inclination and the longitude of the node. As the $\Delta\delta$ may still contain an outstanding constant error, the equations were actually put in the form:

$$\Delta\,\delta = a + b\,\sin(\lambda-\theta) + c\,\cos(\lambda-\theta).$$

These equations were solved substituting in them, 1st the mean results of the years 1895—1898, 2d those of 1899—1902, 3d those of the 8 years combined (in all cases the mean results as derived from the total sums).

In this way we obtained:

	b	c	c corrected
1895—1898	— 0″25	+ 1″11	+ 1″23
1899—1902	+ 0.62	+ 0.63	+ 0.79
1895—1902	+ 0″18	+ 0″86	+ 1″00

The last column contains the values of c corrected for the influence of the 2d term of $\frac{d\delta}{d\lambda}$. The corrections actually applied are its products with the mean corrections of the longitude.

The two partial results do not agree very well, especially those for b, or for the correction of the inclination, and if we compare the corresponding values of $\Delta\,\delta$ from the two four-yeargroups, systematic differences between the two sets are clearly shown. Considering however my results in connection with those of Newcomb there seems to be as yet no sufficient ground to assume a periodic part in the coefficients b and c.

From the 8 years combined we derive:

$$\delta\iota = -\;0''.19$$
$$\iota\,\delta\theta = +\;1''.04$$
$$\delta\theta = +\,11''.5$$

13. Finally we may combine our results with those of NEWCOMB and also with those derived by FRANZ. [1])

For the correction of the inclination we find:

NEWCOMB 1868	$\delta i = -0''15$	weight	3
FRANZ 1892	$+0.37$		1
BAKH. 1899	-0.19		3
Mean result	$\delta i = -0''09$		

The correction of the inclination is thus found to be small.

For the correction of the longitude of the node we find:

NEWCOMB 1868	$\delta \theta = +4''5$	weight	3
FRANZ 1892	$+7.4$		1
BAKH. 1899	$+11.5$		3
Mean result 1885	$\delta \theta = +7''9$.		

As NEWCOMB found for 1710 $\delta \theta = -16''$ (*Researches* p. 273), we obtain:

Correction of the centennial motion $= +14''$.

Physics. — *"On the critical mixing-point of two liquids"*. By J. P. KUENEN. (Communicated by Professor VAN DER WAALS in the meeting of October 31, 1903).

A critical mixing-point of two liquids is in general a point where two coexisting liquids become identical in every respect: it corresponds to a plaitpoint or critical point of the two-liquid plait on VAN DER WAALS's ψ-surface or of its projection in the volume-composition diagram, the socalled saturationcurve for the two liquid phases; the term is used more especially to denote the condition, where the liquids are at the same time in equilibrium with their saturated vapour. In the v—x diagram this condition corresponds to the point of contact between the two-liquid curve in its critical point with the vapour-liquid curve: in this condition a change of temperature will either make the critical point appear outside or disappear inside the vapour-liquid curve. The contact sometimes takes place on the inside of the latter-curve and the two-liquid curve then lies entirely in the metastable and unstable parts of the diagram, or it lies outside in the stable part of the figure. In other cases it is the vapour curve the critical point of which comes into contact with a two-liquid curve, but whatever the case may be, the geometrical conditions are the

[1]) The combined results of FRANZ from the observations at Königsberg and at Gottingen have been considered in my second paper. (*Added* Dec. 1903).