*Citation:*

P.H. Schoute, Plücker's numbers of a curve in Sn, in:
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causes great damage to plantations of *Abies balsamea* in the environs of Guelph in Ontario. Though it may be very probable that the fungus mentioned does not belong to the genus *Trimmatostroma*, yet it appears from DOHERTY's article that it greatly impedes the growth of the trees by choosing their needles as substrate. About the checking of the evil nothing is mentioned by DOHERTY, so that we cannot profit by advice from Ontario. No suffering trees were found at Nunspeet except at "de Groote Bunte".

### EXOSPORINA OUD. n. g.

Fungi expositi vel endogeni, stromate nullo vel parum evoluto, conidiis in catenas stipatas digestis, singulatim secedentibus, homomorphis, continuis, coloratis.

E. Laricis OUD. — Stromatibus amphigenis, expositis, punctiformibus, nigris, catenas conidiorum longiusculas, in placentam convexam arcte condensatas, gerentibus; conidiis primo angulatis, denique globulosis, continuis, 5—6 $\times$ 5 $\mu$, singulatim secedentibus, ferrugineis.

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EXPLANATION OF THE PLATE.

Fig. A. Needle of *Larix decidua;* magnification 10 times; with the black spots of *Exosporina Laricis* OUD.

Fig. B. Hyphae or ribbons, extending over the leaf and in various places grown out to small-celled little disks, from which later the conidia, connected to strings, will arise Magn. $\frac{200}{1}$.

Fig. C. Ripe cushion of strings of conidia, as they would appear on a cross-section. Magn. $\frac{500}{1}$.

Fig. D. Part of such a cushion, enlarged $\frac{1000}{1}$. Each separate string shows a spherical top-cell.

Fig. E. CORDA's picture of *Trimmatostroma Salicis.*

Fig. F. SACCARDO's picture of *Exosporium fructicola.*

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I am much indebted to Mr. C. J. KONING at Bussum, who has been kind enough to draw the plate for me.

**Mathematics.** — "PLÜCKER's *numbers of a curve in $S_n$*" by Prof. P. H. SCHOUTE.

The PLÜCKER's numbers of a curve in the space $S_n$ with $n$ dimensions have been given for the first time by VERONESE (*Math. Annalen*, vol. 19, page 195), yet they have been seldom applied although dating from 1882. This is probably due to the more or less awkward notation which is made use of and which has been adopted i. a in PASCAL's *Repertorium der höheren Mathematik* (Leipzig, Teubner, 1902). In the following lines I intend to give a more concise notation, making it possible to write down the $3(n-1)$ relations between the $3n$ PLÜCKER's quantities in three formulae with an index, which must take the values $1, 2, \ldots, n-1$ successively. In order to make the deduction clear to those, who are not so familiar with polydimensional theories I shall begin by indicating them for the case $n=3$ of our space.

2. As is known the six relations between the nine PLÜCKER's quantities of a skew curve are derived in two triplets from the consideration of two plane curves, the first of which is the central projection of the given skew curve $C$ from any point $O$ on any plane $\alpha$, whilst the second is the section of any plane $\alpha$ with the developable of the tangents to the curve $C$. Let us indicate successively order, rank, class of the curve $C$ by $n$, $r$, $m$ and let us represent as is customary by $(a, b)$, $(g, h)$, $(x, y)$ the three pairs of dualistically related numbers, of which $b$ is the number of stationary points, $h$ the number of apparent nodes, $x$ the order of the nodal curve of the developable; then the sextuples

$$(n_1, m_1, d_1, t_1, k_1, b_1) \quad , \quad (n_2, m_2, d_2, t_2, k_2, b_2)$$

of the quantities (order, class and numbers of nodes, double tangents, cusps and inflexions) characterizing the two plane curves are expressed by the equations

$$\left.\begin{array}{l} n_1 = n \\ m_1 = r \\ d_1 = h \\ t_1 = y \\ k_1 = b \\ b_1 = m \end{array}\right\} \quad , \quad \left.\begin{array}{l} n_2 = r \\ m_2 = m \\ d_2 = x \\ t_2 = g \\ k_2 = n \\ b_2 = a \end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

in the nine characterizing values of $C$; so in connection with the well known PLÜCKER's formulae for a plane curve the two triplets of relations hold good:

$$\left.\begin{array}{r} r = n(n-1) - 2h - 3b \\ n = r(r-1) - 2y - 3m \\ m - b = 3(r-n) \end{array}\right\} \quad , \quad \left.\begin{array}{r} m = r(r-1) - 2x - 3n \\ r = m(m-1) - 2g - 3a \\ a - n = 3(m-r) \end{array}\right\} \quad . \quad (2)$$

If we substitute $r_0, r_1, r_2, r_3, r_4$ for the row of quantities $b, n, r, m, a$

and if we put $d_1, t_1$ and $d_2, t_2$ for $h, y$ and $x, y$ the equations (1) pass into

$$\left.\begin{aligned} n_i &= r_i \\ m_i &= r_{i+1} \\ k_i &= r_{i-1} \\ b_i &= r_{i+2} \end{aligned}\right\} . \; (i = 1, 2) \quad . \; . \; . \; . \; . \; . \; . \quad (1')$$

and identities, whilst the two triplets of equations (2) are united to

$$\left.\begin{aligned} r_{i+1} &= \quad r_i(r_i - 1) - 2\,d_i - 3\,r_{i-1} \\ r_i &= r_{i+1}(r_{i+1} - 1) - 2\,t_i - 3\,r_{i+2} \\ r_{i+2} - r_{i-1} &= 3\,(r_{i+1} - r_i) \end{aligned}\right\} , \; (i = 1, 2) \quad . \; . \quad (2')$$

And now these equations pass into those for the general case as soon as the addition $(i = 1, 2)$ is exchanged for $(i = 1, 2, \ldots, n-1)$.

3. We shall now pass to the general case of a curve $C_n$ of order $n'$, lying in an $S_n$ but not in an $S_{n-1}$ and we remind the readers how here we determine the $3(n-1)$ relations between the $3n$ characterizing quantities. If we take in $S_n$ two non-intersecting spaces $S_p$, $S_{n-p-1}$, and if we project $C_n$ out of $S_{n-p-1}$ on $S_p$, the projection is a curve $C_p$. If we imagine this process to be performed for $p = 2, 3, \ldots, n-1$, we arrive at — the curve $C_n$ included — a series of $n-1$ curves $C_2, C_3, \ldots, C_{n-1}, C_n$. If farthermore $\overset{(p)}{C_2}$ — once more for $p = 2, 3, \ldots, n-1$ — is the section of the locus of the spaces $S_{p-1}$ through $p$ successive points of the curve $C_{p+1}$ with any plane lying in the space $S_{p+1}$ of that curve, we arrive at, — if the plane curve $C_2$ already found above is represented by $\overset{(1)}{C_2}$ —, $n-1$ plane curves $\overset{(1)}{C_2}, \overset{(2)}{C_2}, \ldots \overset{(n-1)}{C_2}$ and these furnish $n-1$ triplets of relations. If the sextuples $(n_i, m_i, d_i, t_i, k_i, b_i)$, $(i = 1, 2, \ldots, n-1)$ represent the characterizing numbers of these plane curves the $3(n-1)$ equations hold good:

$$\left.\begin{aligned} m_i &= n_i\,(n_i - 1) - 2d_i - 3k_i \\ n_i &= m_i\,(m_i - 1) - 2\,t_i - 3b_i \\ b_i - k_i &= 3\,(m_i - n_i) \end{aligned}\right\} , \; (i = 1, 2, \ldots, n-1).$$

By representing the series of $n + 2$ quantities

| $b$ | $n'$ | $r^{(1)}, r^{(2)}, \ldots, r^{(n-2)}$ | $m$ | $a$ |
|---|---|---|---|---|
| number of stationary points, | order, | $n-2$ numbers of rank, | class, | number of stationary spaces $S_{n-1}$ |

of $C_n$ by $r_0, r_1, \ldots, r_n, r_{n+1}$ we find the equations (1′) extended from $(i = 1, 2)$ to $(i = 1, 2, \ldots, n—1)$, which equations cause the above-mentioned $3(n—1)$ equations to pass into the equations (2′), in the same way extended from $(i = 1, 2)$ to $(i = 1, 2, \ldots, n—1)$.

4. According to this notation the system of the $3n$ Plücker's numbers of $C_n$ consists of three groups: a group of $n + 2$ quantities $r$ (numbers of rank), a group of $n—1$ quantities $d$ (numbers of double points), a group of $n—1$ quantities $t$ (numbers of double tangents). We shall now indicate what is the exact signification of those quantities.

*Numbers of rank.* We consider $r_0$, $r_p$, $r_{n-1}$ separately.

By $r_0$ we understand the number of stationary points of $C_n$ i.e. the number of the points through which $n + 1$ successive spaces $S_{n-1}$ through $n$ successive points of the curve pass.

For $p = 1, 2, \ldots, n$ we find that $r_p$ indicates how many spaces $S_{p-1}$ through $p$ successive points of $C_n$ cut any space $S_{n-p}$; of these numbers $r_1$ is the order and $r_n$ the class of $C_n$.

The number of stationary spaces $S_{n-1}$ of $C_n$, i.e. the number of spaces $S_{n-1}$ through $n + 1$ successive points of $C_n$ is indicated by $r_{n+1}$.

*Numbers of double points.* The quantity $d_p$ is the number of double points of the section $C_2$ of the locus of the spaces $S_{p-1}$ through $p$ successive points of $C_{p+1}$ with a plane situated in the space $S_{p+1}$ of that curve. So by returning from the projection $C_{p+1}$ to the given curve $C_n$ we find the following: If we project the single infinite number of spaces $S_{p-1}$ through $p$ successive points of $C_n$ out of any space $S_{n-p-2}$ we find a single infinite number of spaces $S_{n-2}$ and therefore a twofold infinite number of intersections $S_{n-4}$ of two non-successive spaces $S_{n-2}$. The locus of those spaces $S_{n-4}$ is a curved space with $n—2$ dimensions, cut in a certain number of points by any plane. This number of points, at the same time the order of this curved space, is $d_p$.

*Numbers of double tangents.* The quantity $t_p$ is the number of double tangents of $\overset{(p)}{C_2}$. By ascending from $C_{p+1}$ to $C_n$ we arrive at the following: By projecting the single infinite number of spaces $S_p$ through $p+1$ successive points of $C_n$ out of any space $S_{n-p-2}$, we find a single infinite number of spaces $S_{n-1}$ enveloping a curved space of $n—1$ dimensions. The number of double tangents of any plane section of this envelope is $t_p$.

For the rest it is easy to see that the numbers $d_p$ and $t_{n-p}$ refer to quantities dualistically opposite in the space $S_n$.

5. By means of the simple form of the PLÜCKER's formulae we are now enabled to show more clearly that really all curves $\overset{(p)}{C_2}$ belong as they should to the same genus. For this we prove the equality of the genera $g_i$ and $g_{i+1}$ of $\overset{(i)}{C_2}$ and $\overset{(i+1)}{C_2}$

According to the relations (1′) extended to $C_n$ we have

$$2\,g_i = (n_i-1)(n_i-2) - 2\,(d_i+k_i) = (r_i-1)(r_i-2) - 2\,(d_i+r_{i-1})$$

and therefore

$$2\,(g_{i+1}-g_i) = (r^2_{i+1}-r^2_i) - 2\,(d_{i+1}-d_i) - 3\,(r_{i+1}-r_i) - 2\,(r_i \quad r_{i-1})\,..(3).$$

Moreover the first of the three equations (2′) for $i$ and $i+1$ gives by means of subtraction

$$r_{i+2}-r_{i+1} = (r^2_{i+1}-r^2_i) - 2\,(d_{i+1}-d_i) - (r_{i+1}-r_i) - 3\,(r_i-r_{i-1})\,.(4)$$

Thus by subtraction of (4) from (3) we get

$$\begin{aligned} 2\,(g_{i+1}-g_i) &= (r_{i+2}-r_{i+1}) - 2\,(r_{i+1}-r_i) + (r_i-r_{i-1}) \\ &= (r_{i+2}-r_{i-1}) - 3\,(r_{i+1}-r_i) \end{aligned}$$

and from this equation the second member disappears in consequence of the third of the equations (2′).

Let us observe by the way that the numbers of rank $r_0, r_1, r_2, .. r_{n+1}$ of $C_n$ form the first terms of a recurrent series with the third of the equations (2′) as equation of condition and thus — for $x$ as the variable — with $(1-x)^3$ as denominator of the generating fraction.

In order to cause the representation to remain as simple as possible we have supposed the curve $C_n$ to lack all higher singularities. For the influence of the latter we refer to the above-mentioned essay of VERONESE.

The PLÜCKER's formulae given here shall be applied elsewhere to the case of the curve $C_n$ of order $2^{n-1}$ forming in $S_n$ the section of $n-1$ quadratic spaces $Q^2_n$.

**Mathematics.** — *"On systems of conics belonging to involutions on rational curves."* By Prof. JAN DE VRIES.

1. We suppose the points of a rational plane curve $C^n$ to be arranged in the groups of an involution $I^s$, $s \geqq 5$, and bring a conic $C^2$ through each quintuple of points belonging to a selfsame group. The system $[C^2]$ formed in this way has evidently no double right lines, so that $\eta = 0$. So between the characterizing numbers $\mu, \nu, \sigma$ exist the relations $2\,\nu = \mu + \sigma$ and $2\,\mu = \nu$; so we have $= 2\,\mu$ and $\sigma = 3\,\mu$.