Mathematics. — "*On the differential equation of* MONGE" by Prof. W. KAPTEYN.

(Communicated in the meeting of January 30, 1904).

If we suppose that in the differential equation

$$Hr + 2\,Ks + Lt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

H, K, L depend only on p and q, the necessary and sufficient conditions for the existence of two intermediate integrals are the following.

In the first place H, K and L must be proportional to the second differential coefficients of a function θ; thus

$$\frac{H}{\theta_{11}} = \frac{K}{\theta_{12}} = \frac{L}{\theta_{22}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the second place the function θ must satisfy the differential equation of the fourth order

$$2\,D\left(\theta_{22}\frac{\partial^2 D}{\partial p^2} - \theta_{12}\frac{\partial^2 D}{\partial p\,\partial q} + \theta_{11}\frac{\partial^2 D}{\partial q^2}\right) =$$

$$= 3\left[\theta_{22}\left(\frac{\partial D}{\partial p}\right)^2 - 2\,\theta_{12}\frac{\partial D}{\partial p}\frac{\partial D}{\partial q} + \theta_{11}\left(\frac{\partial D}{\partial q}\right)^2\right] \quad . \quad . \quad (3)$$

where

$$D = \sqrt{\theta_{12}^{\,2} - \theta_{11}\,\theta_{22}}.$$

The general integral of this differential equation can be represented by the following formulae:

$$\left.\begin{aligned} p &= \frac{g' - h'}{u - v} \\ q &= \frac{\varphi' - \psi'}{u - v} \\ \theta = 2\int g'\,\varphi''\,du - 2\int h'\,\psi''\,dv &- (g' - h')(\varphi' + \psi') - \\ - \frac{2}{u - v}\Big[(g - h)(\varphi' - \psi') &- (\varphi - \psi)(g' - h')\Big] \end{aligned}\right\} \quad (4)$$

where u and v indicate arbitrary parameters and

$$g = g(u) \quad , \quad h = h(v) \quad , \quad \varphi = \varphi(u) \quad , \quad \psi = \psi(v)$$

four arbitrary functions, and

$$g' = \frac{dg}{du} \quad , \quad g'' = \frac{d^2 g}{du^2} \text{ etc.}$$

If the conditions (2) and (3) are satisfied the equation (1) will possess the two intermediate integrals

$$z - z\,\varphi'' - x\,g'' = \mathfrak{F}(u),$$
$$z - y\,\psi'' - x\,h'' = \mathfrak{F}(v),$$

where $\mathfrak{F}$ denotes an arbitrary function and u and v the functions of p and q, derived from the equations (4).

In the particular case that the function θ satisfies the two members of the equation (3) separately, we have to distinguish two cases according to the double sign in

$$\frac{\partial D}{\partial q} - \frac{\theta_{12} \pm \sqrt{\theta_{12}^2 - \theta_{11}\,\theta_{22}}}{\theta_{11}} \frac{\partial D}{\partial p} = 0.$$

The general integral can be written in both cases respectively

$$\left.\begin{array}{l} p = \dfrac{1}{u} f\,[u\,(v \mp g'')] + r''\,(u) \\ q = v \\ \theta = (2\,u^2\,g'' - 4\,u\,g' + 4\,g \mp u^2\,v)\,p - 2\displaystyle\int (u\,g''' \mp v) u\,p\,d\,u \end{array}\right\}, \;. \quad (5)$$

where $g = g\,(u)$ has the same meaning as before, whilst f and r represent two new arbitrary functions of the arguments placed after these symbols.

In these cases the two intermediate integrals are

$$z \mp y\,(u\,g''' - g'') - x(u\,r''' + r'') = \mathfrak{F}\,(u),$$
$$y + xf'\,[u\,(v \mp g'')] = \mathfrak{F}\,[u\,(v \mp g')],$$

the values of u and v being expressed in p and q with the aid of the formulae (5).

The condition (3) appears, although in a different shape, already in the excellent dissertation of J. Vályi (Klausenburg 1880).

Mathematics. — "*The singularities of the focal curve of a plane general curve touching the line at infinity σ times and passing ε times through each of the imaginary circle points at infinity.*" By Dr. W. A. Versluys. (Communicated by Prof. P. H. Schoute).

(Communicated in the meeting of January 30, 1904).

In "Verhandeling" 5 of the "Kon. Ak. v. W." at Amsterdam Vol. VIII, I have deduced some formulae expressing the singularities of the focal developable and of the focal curve in function of the singularities of a plane curve having no particular position.

In a similar way it is possible to deduce the following formulae expressing the singularities of the focal developable and of the focal curve of a plane curve touching the line at infinity σ times and