theorem I for the polytope C_n out of the one for the polytope A_n. By repeated application we arrive at:

"A $A_{2^q-1}^{(\sqrt{2^{q-2}})}$ of the space S_{2^q-1} can project itself on a definite system of mutually rectangular spaces S_{2^q-1}, S_{2^q-2}, ... S_4, S_2, S_1 respectively according to a $C_{2^q-1}^{(\sqrt{2^{q-3}})}$, two coincided $C_{2^q-2}^{(\sqrt{2^{q-4}})}$, ... 2^{q-3} coincided $C_4^{(\sqrt{2})}$, 2^{q-2} coincided squares $C_2^{(1)}$ and 2^{q-1} coincided linesegments $C_1^{(1)}$".

Mathematics. — "*On symmetric transformation of S_4 in connection with S_r and S_l.*" By Mr. L. E. J. Brouwer. (Communicated by Prof. D. J. Korteweg).

Let us for the present occupy ourselves with a particular case of symmetric transformation — the *reflection*, and let us investigate its influence on S_r and S_l. As S_r and S_l are independent of the choice of a system of axes, we make a suitable choice by selecting the X_4 axis along the axis of reflection. Let us call $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the cosines of direction of a vector before the reflection; $\beta_1, \beta_2, \beta_3, \beta_4$ those after it; let us moreover represent $\alpha_2' \alpha_3'' - \alpha_3' \alpha_2''$ etc. by ξ_{23} etc. and $\beta_2' \beta_3'' - \beta_3' \beta_2''$ etc. by χ_{23} etc. and let us call λ_{23} etc. the coefficients of position of a plane with sense of rotation included before the reflection and μ_{23} etc. those after it. Then:

$$\alpha_1 = \beta_1$$
$$\alpha_2 = \beta_2$$
$$\alpha_3 = \beta_3$$
$$\alpha_4 = -\beta_4$$

$$\xi_{23} = \chi_{23} \qquad \xi_{14} = -\chi_{14}$$
$$\xi_{31} = \chi_{31} \qquad \xi_{24} = -\chi_{24}$$
$$\xi_{12} = \chi_{12} \qquad \xi_{34} = -\chi_{34}$$

$$\sqrt{\xi^2_{23}+\xi^2_{31}+\xi^2_{12}+\xi^2_{14}+\xi^2_{24}+\xi^2_{34}}=\sqrt{\chi^2_{23}+\chi^2_{31}+\chi^2_{12}+\chi^2_{14}+\chi^2_{24}+\chi^2_{34}}.$$

So also.

$$\lambda_{23} = \mu_{23} \qquad \lambda_{14} = -\mu_{14}$$
$$\lambda_{31} = \mu_{31} \qquad \lambda_{24} = -\mu_{24}$$
$$\lambda_{12} = \mu_{12} \qquad \lambda_{34} = -\mu_{34}$$

or:

$$\left.\begin{array}{ll} \lambda_{23} + \lambda_{14} = \mu_{23} - \mu_{14} & \lambda_{23} - \lambda_{14} = \mu_{23} + \mu_{14} \\ \lambda_{31} + \lambda_{24} = \mu_{31} - \mu_{24} & \lambda_{31} - \lambda_{24} = \mu_{31} + \mu_{24} \\ \lambda_{12} + \lambda_{34} = \mu_{12} - \mu_{34} & \lambda_{12} - \lambda_{34} = \mu_{12} + \mu_{34}. \end{array}\right\} \quad (a)$$

Now however

$$\begin{array}{l} \lambda_{23} + \lambda_{14} \\ \lambda_{31} + \lambda_{24} \\ \lambda_{12} + \lambda_{34} \end{array}$$

are the cosines of direction of the representant of the system of planes equiangular to the right with λ with respect to a system of coordinates $OX_r Y_r Z_r$ taken in S_r, as that was defined (These Proceedings Febr. 1904, page 729).

And likewise

$$\begin{array}{l} \lambda_{23} - \lambda_{14} \\ \lambda_{31} - \lambda_{24} \\ \lambda_{12} - \lambda_{34} \end{array}$$

are the cosines of direction of the representant of the system of planes equiangular to the left with λ with respect to a system of coordinates $OX_l Y_l Z_l$ taken in an analogous manner in S_l.

So from the formulae (a) ensues that the effect of a reflection is what we might call a *reciprocal interchange of S_r and S_l*, i.e. a suchlike interchange that every vector of S_l takes the place of that vector of S_r, which has substituted itself for it.

But now an arbitrary symmetric transformation of S_4 can be replaced by a reflection preceded or followed by a double rotation; which is represented by a reciprocal interchange of S_r and S_l preceded or followed by a rotation of S_r and one of S_l; therefore: The arbitrary symmetric transformation of S_4 is represented by an *interchange of S_r and S_l in arbitrary positions.*

Let us now consider that for such an arbitrary interchange of S_r and S_l a system of coordinates α of S_r is placed on a system of coordinates β of S_l whilst that system of coordinates β of S_l itself is placed on a system γ of S_r; then we can replace the interchange by a "reciprocal interchange" placing α on β and β on α, followed by a rotation of S_r, placing α on γ, or also by a rotation of S_r, placing α on γ, followed by a reciprocal interchange, placing γ on β and β on γ.

Consequently we have proved.

"An arbitrary symmetric transformation of S_4 can be replaced by a reflection preceded or followed by a double rotation equiangular to the right and likewise of course by a reflection preceded or followed by a double rotation equiangular to the left."

The plane of rotation of the equiangular double rotation passing through the axis of reflection remains for both parts of the transformation in an unaltered position; it undergoes by the double rotation a congruent transformation and by the reflection a symmetric one.

The plane of rotation of the equiangular double rotation situated in the space perpendicular to the axis of reflection remains also for both parts of the transformation in an unaltered position; it is not transformed at the reflection and undergoes by the double rotation a congruent transformation.

Those two planes of rotation are perpendicular to each other, so that geometrically the wellknown property is proved:

"For symmetric transformation of S_4 about a fixed point one pair of planes remains at its place; and one plane of it is transformed congruently, the other symmetrically."

Physics. — *"On the equations of* CLAUSIUS *and* VAN DER WAALS *for the mean length of path and the number of collisions."* By Dr. PH. KOHNSTAMM. (Communicated by Prof. VAN DER WAALS).

Several of the methods proposed for the derivation of the equation of state, make use of formulae for the mean length of path. It is therefore not to be expected that we shall arrive at undoubted results as to the former, so long as the results as to the latter quantity are not concordant. Now it is generally known that VAN DER WAALS has found for the length of path and the number of collisions in a gas with perfectly hard, perfectly elastic spherical molecules:

$$l = \frac{v-b}{\pi ns^2}\frac{\overline{u}}{\overline{r}} \qquad P = \frac{\pi ns^2}{v-b}\overline{r} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It does not seem to be so generally known, that CLAUSIUS [1]) and in accordance with him JÄGER [2]) and BOLTZMANN [3]), have obtained another result, viz:

$$l = \frac{v}{\pi ns^2}\frac{\overline{u}}{\overline{r}}\frac{1-2\frac{b}{V}}{1-\frac{11}{8}\frac{b}{V}} \qquad P = \frac{\pi ns^2}{v}\overline{r}\frac{1-\frac{11}{8}\frac{b}{v}}{1-2\frac{b}{v}} \quad . \quad . \quad (2)$$

1) Kinetische Theorie der Gase, p. 60.

2) Wien. Sitzungsber. **105**, p. 97.

3) BOLTZMANN Gastheorie, p. 164.