We consider an arbitrary S_3 which intersects the planes of Ω in the axes of rotation of the section; the determination of the ratios of the intensities belonging to them is a well known problem.

If, finally, we notice that between the intensities ω and ω' of a rotation in S_4 and its intersection with a space A the relation $\omega' = \omega \sin (A\omega)$ exists, then in this way the intensities of the rotations about the five associated planes have become known quantities.

Mathematics. — "*On interpolation based on a supposed condition of minimum.*" By J. WEEDER. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN.)

For the reduction of the daily rates of the standard clock in the Leyden Observatory I have developed a method of interpolation, which may perhaps also be profitably used for other investigations.

The following is the problem we have to deal with· a variable quantity, here the correction of the clock, is given for a series of instants, during a long period, with unequal intervals; how can we find an intermediate value of that correction at any moment.

First I tried to solve this problem with the limiting condition that for all the intervals of time which enter into the calculation there is a smallest common divisor, which we take as unit of time.

1. Let S (clock correction) be the variable quantity, and g (rate) the amount by which it increases during a unit of time. Let S_p and S_q, be two successively determined values of S separated by m units of time, then $\frac{S_q - S_p}{m}$ is the average increase per unit; that increase is represented by Q_m. Hence the m quantities $g_1\, g_2 \ldots g_i \ldots g_m$ of the interval considered depend on the relation $\sum_{i=1}^{i=m} g_i = S_q - S_p = m\, Q_m$ and a similar relation exists for each interval between two consecutive determinations of S.

In order to determine the quantities g, I put the condition that the sum of the squares of the differences of the first order for the whole period of observation should be a minimum. This condition of minimum was selected with a view to the special case where we have to interpolate between the clock corrections, but I doubt whether in all cases these interpolated values will be the most probable ones. Leaving aside for the moment these considerations, I go on developing the problem in hand. The quantities which correspond to an interval of m units occur only in the following terms of the sum which according to this condition must be a minimum:

$$(g_1-g_p)^2 + (g_2-g_1)^2 \cdots\cdots + (g_m-g_{m-1})^2 + (g_q-g_m)^2,$$

g_p represents the rate preceding S_p and g_q the rate following S_q.

If to this quantity we add:

$$2k_m(g_1+g_2\cdots\cdots+g_i\cdots\cdots+g_{m-1}+g_m)$$

their sum U will also be a minimum for the same values g_1 to g_m, because $\sum_{i=1}^{i=m} g_i = S_q - S_p = mQ_m$ is constant.

These values $g_1 \ldots g_m$ are found by means of the condition that each derivative of U taken with reference to each variable shall be zero. At the same time we must assign a definite value to k_m in order to satisfy the equation $\sum_{i=1}^{i=m} g_i = mQ_m$.

Thus we obtain the following equations:

$$\left.\begin{array}{l} -g_p + 2g_1 - g_2 + k_m = 0 \\ -g_1 + 2g_2 - g_3 + k_m = 0 \\ \cdots\cdots\cdots\cdots \\ -g_{i-1} + 2g_i - g_{i+1} + k_m = 0 \\ \cdots\cdots\cdots\cdots \\ -g_{m-2} + 2g_{m-1} - g_m + k_m = 0 \\ -g_{m-1} + 2g_m - g_q + k_m = 0 \end{array}\right\} \quad \ldots\ldots \quad (A)$$

Bij taking the sum of these m equations, each multiplied by $i(m-i)$, we eliminate g_1, g_i, g_m, the coefficient of each g_i of that sum being equal to 2; hence the sum of the terms is $2\sum_{i=1}^{i=m} g_i = 2mQ_m$.

The coefficient of k_m is:

$$\sum_{i=1}^{i=m} i(m-i) = \frac{m(m+1)(m+2)}{1.2.3}.$$

Hence the sum of the multiplied equations is:

$$-mg_p + 2mQ_m - mg_q + k_m\frac{m(m+1)(m+2)}{6} = 0$$

whence
$$k_m = \frac{6(g_p+g_q-2Q_m)}{(m+1)(m+2)}.$$

Then we determine the values of g_i by multiplying the m equations by the terms of the following series:

$1(m-i+1),\ 2(m-i+1),\ldots(i-1)(m-i+1),\ i(m-i+1),\ i(m-i),\ldots,\ i.2,\ i.1$

and taking their sum; in the resulting equation all unknown quantities $g_1 \dots g_m$ except g_i are eliminated. That equation got by summation is:

$$-g_p(m-i+1)+(m+1)g_i - i\,g_q + \frac{1}{2}\,i\,(m-i+1)\,(m+1)\,k_m = 0$$

which yields:

$$g_i = \frac{m-i+1}{m+1}\,g_p + \frac{i}{m+1}\,g_q - \frac{i\,(m-i+1)}{2}\,k_m$$

hence
$$g_1 = \frac{m}{m+1}\,g_p + \frac{1}{m+1}\,g_q - \frac{1}{2}\,m\,k_m$$

and
$$g_m = \frac{1}{m+1}\,g_p + \frac{m}{m+1}\,g_q - \frac{1}{2}\,m\,k_m\,.$$

The quantities g_p and g_q are still unknown and depend on the quantities Q of the neighbouring intervals, they may be derived from them by means of successive approximation.

It gives some advantage to determine $\frac{1}{2}(g_p+g_1)$ and $\frac{1}{2}(g_m+g_q)$ by approximations, because then we shall have to approximate only one quantity for each S. The approximation may be made in the following way: we put $\frac{1}{2}(g_p+g_1)=c_p$ and $\frac{1}{2}(g_m+g_q)=c_q$, then we obtain:

$$k_m = \frac{6}{m^2+2}\,(c_p + c_q - 2\,Q_m)$$

$$g_p = -\frac{3m}{m^2+2}\,Q_m + \left(1+\frac{2m^2+1}{m(m^2+2)}\right)c_p + \frac{m^2-1}{m(m^2+2)}\,c_q$$

$$g_q = -\frac{3m}{m^2+2}\,Q_m + \frac{m^2-1}{m(m^2+2)}\,c_p + \left(1+\frac{2m^2+1}{m(m^2+2)}\right)c_q\,.$$

For the next interval of n units of time between the determinations S_q and S_r we have the following equation:

$$g_m = -\frac{3n}{n^2+2}\,Q_n + \left(1+\frac{2n^2+1}{n(n^2+2)}\right)c_q + \frac{n^2-1}{n(n^2+2)}\,c_r\,.$$

As $g_q + g_m = 2\,c_q$ we obtain when finding the summation of the two last equations a recurrent equation containing 3 consecutive quantities c, so that c_q can be expressed in c_p and c_r. This equation can also be written thus:

$$\left\{\frac{2m^2+1}{m(m^2+2)}+\frac{2n^2+1}{n(n^2+2)}\right\}c_q=-\frac{m^2-1}{m(m^2+2)}c_p+\frac{3m}{m^2+2}Q_m+$$
$$+\frac{3n}{n^2+2}Q_n-\frac{n^2-1}{n(n^2+2)}c_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

For the first interval considered here the first of the equations (A) is $g_1-g_2+k_m=0$. This equation may also be written in the general form by putting $-g_1+2g_1-g_2+k_m=0$, thus assuming that the value of g preceding g_1 and c belonging to the first observation are both equal to g_1. In the same way c belonging to the last observation is equal to the last g of the last interval. Between each three consecutive quantities c, therefore, a relation exists of the form (B) and two other equations are added to the beginning and to the end of this series, each containing only two values c derived from the formulae for g_p and g_q. Let c_a and c_b be the first two and c_y and c_z the last two quantities c, then we obtain by substituting c_a for $g_p=c_p$ and c_b for c_q the first condition, and by substituting c_z for $g_q=c_q$ and c_y for c_p the last condition of the series which determine the values c.

If the lengths of the limiting intervals are represented by μ and ν these equations are:

$$(2\mu^2+1)\,c_a=+\,3\mu^2Q_\mu-(\mu^2-1)\,c_b$$
$$(2\nu^2+1)\,c_z=+\,3\nu^2Q_\nu-(\nu^2-1)\,c_y$$

The series (B) and these two equations determine all the quantities c. If we solve them by approximation our purpose is soon gained; we assume to the first approximation $c_q=\dfrac{nQ_m+mQ_n}{m+n}$ and c_a and c_z equal to the values of Q of the first and the last interval respectively. From the equations (B) we derive the first corrections $\Delta_1 c_p$, $\Delta_1 c_q$, etc. and $\Delta_2 c_q$ is derived from the formula:

$$\left\{\frac{2m^2+1}{m(m^2+2)}+\frac{2n^2+1}{n(n^2+2)}\right\}\Delta_2 c_q=-\frac{m^2-1}{m(m^2+2)}\Delta_1 c_p-\frac{n^2-1}{n(n^2+2)}\Delta_1 c_r.$$

In this interpolation we determine g_i and S_i of an interval of m units according to the formulae:

$$g_i=\left(\frac{1}{4}m\,k_m+c_p+\frac{c_p-c_q}{2m}\right)-\left(\frac{m+1}{2}k_m+\frac{c_p-c_q}{m}\right)i+\frac{1}{2}k_m\,i^2$$
$$S_i=S_p+\left(c_p-\frac{1}{6}k_m\right)i-\left(\frac{m}{4}k_m+\frac{c_p-c_q}{2m}\right)i^2+\frac{1}{6}k_m\,i^3.$$

2. In the previous section the observed and the interpolated quantities S, occurring in the problem discussed, form a series of

discrete values corresponding to an arithmetical series of the argument; now I will remove the restriction of commensurable arguments and will make this mode of interpolation applicable to a continuous varying quantity and an arbitrary argument by putting for the ratio of that series the infinitely small value dt. The condition of minimum then becomes $\int_a^z \left(\frac{d^2S}{dt^2}\right)^2 dt = \min.$

The formulae for this continuous interpolation may be derived independently, but it is shorter to derive them from the corresponding formulae of the discrete interpolation developed above. For the present I shall put for the lengths of the intervals between which we have to interpolate m' and n', for the derived values $\frac{dS}{dt}$ of the interpolated function g', to distinguish them from the letters we have used in the former problem.

Instead of m and n we have $\frac{m'}{dt}$ and $\frac{n'}{dt}$; for c_p, c_q, c_r we must substitute the quantities $g'_p\,dt$, $g'_q\,dt$, $g'_r\,dt$, and for Q_m and Q_n the quantities $\frac{S_q-S_p}{m'}\,dt$ and $\frac{S_r-S_q}{n'}\,dt$ or $Q_{m'}\,dt$ and $Q_{n'}\,dt$.

After dividing the relations (B) by dt^2 and omitting the infinitely small values we have:

$$\left(\frac{2}{m'}+\frac{2}{n'}\right) g'_q = -\frac{1}{m'} g'_p + \frac{3}{m'} Q_{m'} + \frac{3}{n'} Q_{n'} - \frac{1}{n'} g'_r$$

from which, after dropping the accents, we get:

$$g_q = \frac{nQ_m + mQ_n}{m+n} + \frac{n(Q_m - g_p)}{2\,(m+n)} + \frac{m(Q_n - g_r)}{2\,(m+n)}. \quad . \quad . \quad . \qquad (C)$$

to which we must add as first and last equations:

$$g_a = Q_\mu + \frac{Q_\mu - g_b}{2} \quad \text{and} \quad g_z = Q_\nu + \frac{Q_\nu - g_y}{2}$$

For k_m we substitute $\frac{6}{m'^2}(g'_p + g'_q - 2\,Q_{m'})\,(dt)^2$; for i we substitute $\frac{t}{dt}$, if t represents the time between the last preceding observation at the moment for which we interpolate. These substitutions in the formula for S_i yield a formula for S_t, which, after the omission of infinitely small values and accents, is:

$$S_t = S_p + g_p t - \left[\frac{3}{2m}(g_p + g_q - 2\,Q_m) + \frac{g_p - g_q}{2m}\right] t^2 + \frac{g_p + g_q - 2Q_m}{m^2} t^3$$

By substituting in the above formula $m-t'$ for t, we obtain for S_t a formula developed according to the ascending powers of t', the interval between the moment for which we interpolate and the moment of the next observation. It is simpler to find the same formula by imagining the interpolation to be made in the inverse direction, so that the quantities g and Q change signs and the indices p and q change places. Hence:

$$S_{m-t'} = S_t = S_q - g_q t' + \left[\frac{3}{2m}(g_p + g_q - 2\,Q_m) - \frac{g_p - g_q}{2m}\right] t'^2 - \frac{g_p + g_q - 2\,Q_m}{m^2} t'^3.$$

For S_t to be interpolated in the following interval we use:

$$S_t = S_q + g_q t - \left[\frac{3}{2n}(g_q + g_r - 2Q_n) + \frac{g_q - g_r}{2n}\right] t^2 + \frac{g_q + g_r - 2Q_n}{n^2} t^3.$$

Therefore the formulae on either side of each observation are different. If in the latter formula t is negative and $-t'$ is substituted for it, the resulting formula differs from the preceding one only in the coefficients of the terms of the 3rd degree. The coefficients of the terms of the 2nd degree have become equal by satisfying the relation (C).

Therefore we also obtain the interpolated function if, by starting from a value (S_q) derived from observation, we represent the values of S_{-t} and S_{+t} for the moments between that observation and the next preceding one and those between that observation and the next following one by the formulae:

$$S_{-t} = S_q - g_q t + c_q t^2 - e_m t^3 \quad \text{and} \quad S_{+t} = S_q + g_q t + c_q t^2 + e_n t^3.$$

Taking this as basis, we find:

$$c_q = \frac{+g_p - 3Q_m + 2g_q}{m} = \frac{-2g_q + 3Q_n - g_r}{n} \quad e_m = \frac{g_p + g_q - 2\,Q_m}{m^2} \quad e_n = \frac{g_q + g_r - 2\,Q_n}{n^2}$$

The integral $\int_a^z \left(\frac{d^2S}{dt^2}\right)^2 dt$, which becomes a minimum through this nterpolation, is equal to the sum of the integrals between two consecutive observations, and each of these integrals can be expressed in the coefficients of the interval in the following manner:

$$I_n = \int_q^r \left(\frac{d^2S}{dt^2}\right)^2 dt = \frac{3(g_q + g_r - 2Q_n)^2}{n} + \frac{(g_q - g_r)^2}{n} = 3\,n^3 e_n^2 + \frac{(g_q - g_r)^2}{n}$$

or:

$$I_n = \frac{4}{3} n\,(c_q^2 + c_q\,c_r + c_r^2).$$

For the total integral ΣI_n we can also derive a simple form by integrating partially:

$$\int_a^z \left(\frac{d^2S}{dt^2}\right)^2 dt = \left[\frac{dS}{dt}\frac{d^2S}{dt^2}\right]_a^z - \int_a^z \frac{dS}{dt}\frac{d^3S}{dt^3}\, dt.$$

For the first moment a and the last moment z, $\frac{d^2S}{dt^2} = 0$, as follows from the first and the last equations belonging to (C). For each interval between two observations $\frac{d^3S}{dt^3}$ is a constant quantity. Hence we find:

$$I = \Sigma\, 6\, e_n (S_q - S_r)$$

where the summation extends over all the intervals between the observations. We can easily find a simple expression for the differential quotient of I according to each of the observed values, which may be useful when we want not only to interpolate for an intermediate moment but when at the same time we have to determine the most probable values of the observed quantities. For then the difficulty presents itself how to find the best method for diminishing the amount of the minimum value I by applying corrections to the observations, of which corrections the mean value is known.

In doing so heed must be taken that these corrections, being errors of observation, shall satisfy the law which determines their probabilities as functions of their magnitudes.

I have not yet reached a satisfactory solution of this problem. The following remarks, however, on this subject seemed important enough to be communicated.

3. Let L_p, L_q, L_r be the observed quantities, free from errors of observation, and f_p, f_q, f_r the errors themselves.

If we have developed the interpolation by means of the quantities L and f separately, we obtain the formulae:

$$L_t = L_q + G_q t + C_q t^2 + E_n t^3$$
$$f_t = f_q + \beta_q t + \gamma_q t^2 + \varepsilon_n t^3 .$$

By means of the summation of these two formulae we get:

$$S_t = S_q + g_q t + c_q t^2 + e_n t^3.$$

If we apply a partial integration to $\int_a^z \frac{d^2L}{dt^2} \cdot \frac{d^2f}{dt^2}\, dt$, we get:

$$\left[\frac{d^2L}{dt^2}\frac{df}{dt}\right]_a^z - \int_a^z \frac{d^3L}{dt^3}\frac{df}{dt}\, dt$$

or $$\left[\frac{dL}{dt}\,\frac{d^2f}{dt^2}\right]_a^z - \int_a^z \frac{dL}{dt}\,\frac{d^3f}{dt^3}\,dt$$

In either case the integrated parts are equal to 0, because at the beginning and end $\frac{d^2L}{dt^2}$ and $\frac{d^2f}{dt^2}$ are zero.

In this way we find the relation:

$$\Sigma\, E_n\,(f_q-f_r) = \Sigma\, \varepsilon_n\,(L_q-L_r).$$

In the same way we find the relation:

$$\Sigma\, e_n\,(f_q-f_r) = \Sigma\, \varepsilon_n\,(S_q-S_r).$$

By applying the corrections $-f$, the minimum I_S becomes the minimum $I_L = I_S - f$.

$$I_S - f = \Sigma\, 6\,(e_n-\varepsilon_n)\,(S_q-f_q-S_r+f_r) =$$
$$= \Sigma\, 6\, e_n\,(S_q-S_r) - \Sigma\, 6\,\varepsilon_n\,(S_q-S_r) - \Sigma\, 6\, e_n\,(f_q-f_r) + \Sigma\, 6\,\varepsilon_n\,(f_q-f_r)$$

which expression by means of the latter relation may be reduced to:

$$I_S - f = I_S - 12\,\Sigma\, e_n\,(f_q-f_r) + \Sigma\, 6\,\varepsilon_n\,(f_q-f_r).$$

For infinitely small values f, the last term in the expression given above becomes of the order f^2 so that we find $\frac{\partial I_S}{\partial S_q} = 12(e_n-e_m)$. This result enables us to determine the set of small corrections, which, when applied to the quantities S, diminish I_S by the greatest amount. These corrections will be proportional to the abrupt changes of $\frac{d^3S}{dt^3}$.

The variations in the interpolation coefficients g, c, e, resulting from these corrections are found by repeating the interpolation, with this sole difference that for the observed quantities S we substitute the abrupt changes of $\frac{d^3S}{dt^3}$.

As a rule a set of corrections of this kind will not show the character of the errors of observation and therefore be dissimilar to the set of errors which actually exist in the observed quantities S. We may also determine a limit which should not be passed in the rectification.

If the quantities f represent the real errors, we have:

$$I_S = I_L + \Sigma\, 12\, E_n\,(f_q-f_r) + \Sigma\, 6\,\varepsilon_n\,(f_q-f_r)$$

The coefficients E of the interpolation formula between the correct quantities S and the errors f being as a rule entirely independent, we must assume that in $\Sigma\, 12\, E_n\,(f_q-f_r)$ the positive and negative terms neutralize each other for the greater part. Hence the difference $I_S - I_L$ does not exceed $\Sigma\, 6\, \varepsilon_n\, (f_q - f_r)$, the value of which depends only on the errors and the lengths of the intervals; the mean value of this expression for every possible distribution of the errors of the observations may be derived from the mean error of those observations.

This is the utmost limit to which by means of corrections to the observed quantities S we can diminish I_S, lest the interpolation curve found should assume a less sinuous form than would be probable with regard to the results of the observations and their precision.

Here follows an example of the computation.

The annexed table contains the interpolation coefficients of a part (period 1882 June 8 to August 30), taken from a longer series of observed rates of the clock Hohwü 17. Therefore the coefficients at the limits of this period are not in accordance with the boundary-conditions supplying the formula (C).

We compute the interpolated clock corrections by means of the formula:

$$S_t = S_q + t\left(g_q + n\, c_q \frac{t}{n} + n^2\, e_n \frac{t^2}{n^2}\right)$$

S_q is the clock correction of the last preceding observation and the coefficients g_q, nc_q, n^2e_n are given in the columns 5, 6 and 7; they are expressed in the unit $0^s.001$. The values g_q and nc_q to be used are placed a little above the horizontal line corresponding to the length of the interval expressed in days, which interval contains the moment t for which we interpolate. Because of its connection with the constant derivative of the third order of the interpolation curve within each interval, the coefficient n^2e_n for each interval has been placed on the horizontal line of that interval.

The 8th column contains the coefficients e and the 9th their differences σ by passing from one interval to the other. For each of these differences I have calculated the variation $\Delta\sigma_q$ of a given σ_q, as the corresponding correction of the clock S_q increases by $+\,0^s.100$ while the other corrections remain unmodified; they are given in the 10th column. By the increase $\Delta S_q = -\frac{\sigma_q}{\Delta\sigma_q} \times 0^s.100$ the difference σ_q becomes zero, so that by means of this increase we obtain the same result as if in the determination of the interpolation curve we had omitted the observation S_q. Hence the correction of the clock S_q derived from this interpolation is equal to the observed

Duration of the intervals in days m_1 n.	Mean daily rates Q.	$\frac{mQ_n+nQ_m}{m+n}$	Correction term $\frac{m(Q_n-q_r)+n(Q_m-q_p)}{2(m+n)}$	g. coefficient of t.	nc. c = coefficient of t^2.	n^2e. e = coefficient of t^3.	e.	$\sigma = e_n - e_m$	$\Delta\sigma$ for a variation $\Delta S = +0^s.100$.	Obs.—Comp. $= \frac{\sigma}{\Delta\sigma} \times 0^s.100$.
		118	−03	115	+ 63			− 7.0	+ 5.4	s −0.13
4	135					−43	− 2 7			
		110	+02	112	− 62			+ 4.9	+ 4.0	+0.12
4	086					+36	+ 2.2			
		106	−10	096	+ 44			− 3.1	+ 3.9	−0.08
4	126					−14	− 0.9			
		126	+16	142	+ 04			− 0.3	+ 3.7	−0.01
4	126					−20	− 1.2			
		102	−12	090	− 68			+ 3.2	+ 3.0	+0.11
5	073					+51	+ 2.0			
		108	−01	107	+ 52			− 5.3	+ 4.1	−0.13
3	129					−30	− 3.3			
		122	−01	121	− 94			+ 4 9	+ 3.3	+0.15
7	104					+77	+ 1.6			
		156	+08	164	+ 18			−20 6	+ 26.0	−0.08
1	163					−19	−19.			
		143	00	143	−150			+23.4	+ 29.3	+0.08
4	064					+71	+ 4.4			
		081	−25	056	+ 46			− 5.3	+ 6.5	−0.08
3	094					−08	− 0.9			
		110	+14	124	+ 32			− 0.6	+ 5.6	−0.01
4	132					−24	− 1.5			
		113	+03	116	− 50			+ 2.5	+ 3.0	+0.08
5	090					+24	+ 1.0			
		089	−01	088	+ 14			− 2 4	+ 4.2	−0.06
3	089					−13	− 1.4			
		082	−05	077	− 45			+ 2 9	+ 4.3	+0.07
5	070					+38	+ 1.5			
		095	+06	101	+ 43			− 5.3	+ 4.6	−0.12
3	110					−34	− 3.8			
		093	−08	085	− 78			+ 7.7	+ 6.4	+0.12
4	070					+63	+ 3.9			
		113	+05	118	+ 28			−25.9	+ 57.0	−0.05
1	124					−22	−22.			
		106	+02	108	− 38			+39.	+167.	+0.02
1	087					+17	+17.			
		090	−07	083	+ 31			−21.2	+ 85.0	−0.03
2	097					−17	− 4.2			
		101	−07	094	− 28			+ 8.8	+ 21.9	+0.04
3	107					+41	+ 4.6			
		153	+08	161	+ 33			−29.6	+ 57.8	−0.05
1	160					−25	−25.			
		148	+04	152	− 84			+34.2	+114.	+0.03
2	105					+37	+ 9.2			
		106	−11	095	+ 81			−11.0	+ 10.6	−0.10
6	110					−66	− 1.8			
		054	+05	059	− 77			+ 3.9	+ 2.6	+0.15
4	016					+34	+ 2.1			
		021	−14	007	+ 25			− 2.5	+ 3.8	−0.07

S_q diminished by $\frac{\sigma_q}{\Delta\sigma_q} \times 0^s.100$. These differences Obs.—Comp. given in seconds of time, are contained in the 11th column.

From the developed formulae I derived for these 24 intervals the value $I_S = \Sigma \frac{4}{3} n (c_q^2 + c_q c_r + c_r^2) = 69500$, while for all the different manners of distribution of the errors of observation the mean of all the values $I_f = \Sigma\, 6\, \varepsilon_n (f_q - f_r)$, which values depend only on the magnitude of the errors and on their distribution is equal to 30500. In the computation the mean error of the observations has been put $0^s.028$, which value must be regarded as the smallest that can be assumed on the strength of other investigations. Therefore the sinuosity of the interpolation curve must be ascribed for a great part to errors of observation.

(December 24, 1902).