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**Mathematics.** — *"On the geometrical representation of the motion of variable systems".* By Prof. J. CARDINAAL.

1. In two communications [1]) some theorems have been developed by me, relating to the motion of variable systems. Also in this subdivision of the doctrine of motion the method of the geometrical representation occurring so frequently in Mathematics can be applied. The following communication has in view to mention some particulars on this subject. The representation in question is treated [2]) by R. STURM. From this treatise I derive the short summary, which must needs appear here as an introduction to the subject.

2. In the quoted considerations two complexes of rays played an important part, namely the tetrahedral complex formed by the directions of the velocities of the points of the moving system and the rays of a focal system belonging to it; the latter consists for the motion of an invariable system of the normals of the trajectories of the points and for a projectively variable system of rays whose construction took a great part of the considerations. The purpose must be to obtain a simultaneous representation of complex and focal system; it will prove desirable to give the foremost place to the representation of the focal system.

3. Let thus be given the focal system $A$ situated in the space $\Sigma$. According to the method of SYLVESTER let us suppose two planes $\xi$ and $\xi'$ with two projective pencils of rays situated in them with their vertices $X'$ and $X$ situated on the line of intersection $\xi\xi' \equiv x$,

1) Proceedings of the Kon. Akad. van Wetensch., section of science, vol. IV, pages 489 and 588.

2) Die Gebilde ersten und zweiten Grades der Liniengeometrie, I, p. 257.

$x$ being an homologous ray of both pencils. The rays of $A$ are the transversals of two homologous rays of $(X\xi')$ and $(X'\xi)$.

Let us now take two sheaves of rays in the space $\Sigma_1$ with the vertices $X_1$ and $X'_1$ and establish a projective correspondence between these sheaves and the pointfields $\xi$ and $\xi'$, in such a way that the pencil of planes through the axis $X_1X'_1$ is homologous to the pencils $(X\xi')$ and $(X'\xi)$. Let $l$ be a ray of $A$, cutting two homologous rays of $(X\xi')$ and $(X'\xi)$, to which in the homologous plane $\lambda_1$ a ray $l_1$ out of $X_1$ and a ray $l'_1$ out of $X'_1$ correspond; $l_1$ and $l'_1$ intersect each other in a point $L_1$. This point is homologous to the ray $l$. So a projective correspondence is established between the points of the space $\Sigma_1$ and the rays of the focal system $A$.

As is the case with every representation, also here the knowledge of its principal curve cannot be dispensed with. It is a conic $X_1{}^2$ through the points $X_1$ and $X'_1$ situated in a plane $\xi_1$. Its points are homologous to the pencils of rays of $A$ situated in planes through $x$. The plane $\xi_1$ (principal plane) itself is homologous to $x$.

To an arbitrary pencil of rays of $A$ a right line corresponds cutting $X_1{}^2$, to a hyperboloidic system of focal rays a conic having two points in common with $X_1{}^2$, to a linear congruence belonging to $A$ a quadratic surface through $X_1{}^2$.

4. Let a projectively variable moving spacial system be given; let as before $PQRS$ be the tetrahedron of coincidence of two successive positions and let the corresponding focal system $A$ be determined by $PQ$ and $RS$ as conjugate polars and the conic $K^2$ touching $PR$ and $PS$ in $R$ and $S$. According to the indicated method the focal system can be represented in the space $\Sigma_1$; for the tetrahedral complex of the directions of the velocities, however, we need another representation, which can be taken in such a way that the same principal curve is retained; we shall succeed in this if we do not represent the complex itself, but its section with the focal system $A$. This gives rise to a congruence (2,2) which we shall first investigate more closely.

5. Let $A$ be an arbitrary point, $\alpha$ its focal plane; at the same time $A$ is the vertex of a quadratic cone, geometrical locus of the directions of the velocities through $A$, but of which only one is the direction of velocity of $A$ itself. This cone will cut in general $\alpha$ into two rays belonging to the congruence (2,2); in this way we can construct the whole congruence. By this we have determined the construction, but not the geometrical character of the congruence; this can be done in the following manner:

Let the direction of the velocity $a$ of a point $A$ intersect the plane of coincidence $PRS$ in $L$; now the focal plane of $A$ intersects this plane in the polar $p$ of $L$ with respect to the conic $K^2$. The rays of the complex, at the same time rays of the conjugate focal system, are situated in the focal plane $\alpha$ of $A$; from this ensues that these rays intersect the plane $PRS$ in two coincident points, at the same time conjugate with respect to $K^2$; so these rays will intersect $K^2$ and now ensues the theorem:

"The rays of the congruence (2,2), which is the section of the complex with the focal system, have a point in common with the conic $K^2$; so they are found as rays of $A$ cutting $K^2$."

So the congruence (2,2) arising from this belongs to those congruences, not possessing a focal surface, but a singular or double curve [1]), geometrical locus of the first series of foci of the congruence.

6 The congruence can be constructed as a whole out of points of the conic $K^2$; for these points have the property of being the points of intersection not only of two but of a whole pencil of rays of the congruence (2,2), situated in the focal planes belonging to each of the points. These focal planes envelop a quadratic cone $P^2$, with the vertex $P$; so the congruence must touch the cone. From this ensues the following construction:

"Let a point $A$ be taken on $K^2$, the focal ray $PA$ be drawn, cutting $K^2$ for the second time in $A'$. Let the two tangential planes to $P^2$ be brought through $PA$; each of these planes contains a pencil of rays of the congruence, the vertex of one pencil being $A$, of the other $A'$."

7. We now proceed by giving some visible properties of the congruence (2,2).

*a.* The two foci of each ray are the points of intersection with $K^2$ and the point of contact with $P^2$. The focal surface of points becomes $P^2$; the focal surface of tangential planes consists of the tangential planes of $K^2$.

*b.* All rays of the congruence (2,2) belonging to a congruence of rays (1,1) of $A$ cut two conjugate polars of $A$, and cutting at the same time $K^2$ they form a ruled surface of order four with a simple conic and two double lines.

*c.* The rays of the congruence (2,2), lying on a hyperboloid of

1) Congruences of this type are ranged in the "Index du répertoire bibliographique des sciences mathématiques" under $N^2 1 e \alpha$ and placed by R. STURM in a separate division; see "Liniengeometrie", II, p. 323.

$A$, pass through the points of intersection of the latter with $K^2$; so they are four in number.

*d.* Let $K^2$ be real and let $P$ be situated within $K^2$; all focal rays through $P$, the focal point of plane $PRS$, now cut $K^2$; so all pencils of rays are real. If $P$ lies outside $K^2$ two tangents out of $P$ can be drawn to $K^2$; these tangents are the lines of intersection of the cone $P^2$ with plane $PRS$. The planes touching $P^2$ according to these lines of intersection are focal planes, in which two pencils of rays have coincided; rays through $P$, not cutting $K^2$, give rise to imaginary pencils of rays of the congruence (2,2). Further ensues from this:

"If $K^2$ is real and all the vertices of the tetrahedron of coincidence likewise are real, the congruence (2,2) is built up of real and imaginary pencils of rays, where as a transition two are double ones; if $K^2$ is real but the vertices $R$ and $S$ are imaginary, all the pencils are real."

*e.* The cases in which $K^2$ is imaginary, or also those in which all the vertices of the tetrahedron of coincidence are imaginary, do not give real congruences; so they are not under consideration.

8. We now pass to the representation of the congruence (2,2) by which the image is obtained of the connection of focal system and tetrahedral complex.

*a.* The congruence containing $\infty$ pencils of rays which are represented in $\Sigma_1$ by straight lines having a point in common with $X_1^2$, the whole congruence is represented by a ruled surface passing through $X_1^2$. To a straight line $l_1$ in $\Sigma_1$ a hyperboloidic system of focal rays corresponds, which has four points in common with $K^2$; so it contains four rays of the congruence and the representing surface $S_1^4$ of the congruence (2,2) is a ruled surface of order four.

*b.* An arbitrary pencil of focal rays of $A$ contains two rays of the congruence; the straight line in $\Sigma_1$ corresponding to them cutting $X_1^2$ has another two points in common with $S_1^4$; so $X_1^2$ is a double conic of $S_1^4$.

*c.* To the pencil of rays in $\Sigma$ with $P$ as vertex and $PRS$ as plane a straight line $p_1$ in $\Sigma_1$ corresponds, cutting $X_1^2$. Each ray of the pencil $P/PRS$ belonging to two pencils of rays whose vertices are points of intersection with $K^2$, in all points of $p_1$ two generators of $S_1$ concur; from this follows that $S_1^4$ is a ruled surface having as double curve a conic with a straight line cutting it; with this the type of $S_1^4$ has been established.

9. A closer acquaintance with the form of $S_1^4$ is obtained by tracing the pinchpoints on the double curve; there can be two of them on $p_1$ and two on $X_1^2$. Those of $p_1$ depend on the position of $P$ with respect to $K^2$.

*a.* Let $P$ be outside $K^2$. When a ray through $P$ cuts $K^2$ in two points, we get two pencils of rays of the congruence, to which two real generators of $S_1^4$ correspond, concurring in a point of $p_1$. For the tangential lines out of $P$ to $K^2$ these two generators coincide, so the point of $S_1^4$, from which they are drawn is a pinchpoint; so for this position there are two real pinchpoints on $p_1$; from this ensues:

"If $P$ lies outside $K^2$, $p_1$ has one part appearing as double line and another which is isolated; two pinchpoints separate these two parts."

*b.* Let $P$ lie within $K^2$. All focal rays through $P$ cut $K^2$; there are no tangents to $K^2$, so there are no pinchpoints on $p_1$. So the double line $p_1$ is in its whole length really double line.

Besides the pinchpoints on $p_1$ the surface $S_1^4$ has also pinchpoints on $X_1^2$. To find these we must keep in view that the points on $X_1^2$ correspond to the pencils of rays whose vertices lie on $XX' \equiv x$, which are thus situated in planes through $x$. Let $\gamma$ be a plane through $x$ and $C$ its focal point; the pencil of rays $(C\gamma)$ has two rays cutting $K^2$ viz. the two rays connecting $C$ and the points of intersection $B$ and $B'$ of $\gamma$ and $K^2$. These two rays are represented in $\Sigma_1$ by a single point $B_1$ of $X_1^2$. Now $CB$ belongs still to another pencil of focal rays, viz. to the pencil whose vertex is $B$ and whose plane is the plane $CBP \equiv \beta$. The latter pencil belongs to the congruence (2,2) and is thus represented by a straight line through $B_1$ lying on $S_1^4$. In a similar way it appears that also a second straight line of $S_1^4$ passes through $B_1$, namely the one which is represented by the pencil of rays $(B'\beta')$ lying in plane $CB'P$. Now again two principal cases may occur:

*a.* $x$ cuts the plane $PRS$ in a point $T$ outside $K^2$. The pencil of rays $T$ lying in this plane has rays cutting $K^2$ in two points, touching $K^2$ or having two imaginary points in common with $K^2$. In this case these are parts of $X_1^2$ through which two generators of $S_1^4$ pass, which have thus to be regarded as points of a double curve, and parts which are isolated; the transition is formed by two pinchpoints, through which two coinciding generators pass; and these last correspond to the pencils of rays, having their vertices on the tangents drawn from $T$ to $K^2$.

*b.* The above mentioned point of intersection $T$ lies within $X_1^2$. All rays through $T$ cut $K^2$; through each point of $X_1^2$ two generators pass, so the whole conic $X_1^2$ is a double curve.

10. Among the particular sections of $S_1^4$ the conics of this surface come into account. These conics have two points in common with $X_1^2$; so (3) to these must correspond in $\Sigma$ hyperboloidic systems of focal-rays of $A$. These can be constructed in the following way:

Let again a point $A$ be taken on $K^2$, its focal plane $\alpha$ be determined, moreover the second point of intersection $A'$ of $\alpha$ with $K^2$ and the focal plane $\alpha'$ of $A'$. If now a pencil of rays be drawn in $\alpha'$ through $A$ (which rays are not focal rays) and likewise through $A'$ in $\alpha$, the pencils $(A, \alpha')$, $(A', \alpha)$ consist of conjugate polars of $A$ between which a projective correspondence is established by means of the focal rays. In connection with $X_1^2$ each pair of conjugate polars causes a hyperboloidic system of focal rays to appear. These two pencils generate them all, so their number is $\infty$.

11. Finally a few particular cases ask for our attention.

*a.* The line of intersection $x$ cuts the plane $PRS$ in a point of the tangent plane $PR$. The pencil of focal rays in the plane $PR$ has as vertex this point of intersection; to this pencil corresponds a pinchpoint on $X_1^2$, but at the same time this pencil of rays has moreover a ray in common with the pencil of rays in the focal plane of the point $R$; so the obtained pinchpoint is at the same time a point of $p_1$; from this follows that in the point of intersection of $X_1^2$ and $p_1$ two pinchpoints have coincided; so through this point only a single generator of $S_1^4$ can be drawn.

*b.* Application to the motion of an invariable system. In this case $K^2$ is imaginary (the imaginary circle in the plane at infinity); so the congruence (2,2) consists entirely of imaginary rays. The pencil of rays $P/PRS$, however, remains real; so the representation in $\Sigma$ becomes an imaginary ruled surface $S_1^4$ with real double curve consisting of a straight line and a conic intersecting it. The same observation can be made for other cases where $K^2$ becomes imaginary.

*c.* Another particular case occurs when the ray $XX' \equiv x$ is taken in such a way that it cuts the conic $K_1^2$; by doing so the character of the congruence does not change, but its representation does. If we now consider a pencil of rays in a plane brought through $x$, it is apparent that always one of the two rays of congruence to $K^2$ coincides with $x$. Of the two rays cutting in $\Sigma_1$ the double conic $X_1^2$ only one is situated on $S_1^4$, the other one passes into a ray situated in $\xi_1$; from this follows:

"When the focal ray $x$ cuts the conic $K^2$ the surface $S_1^4$ breaks up into $\xi_1$ and a cubic ruled surface $S_1^3$ of which $p_1$ is a double line; so this gives a simpler representation of the congruence (2.2)