description of the strongly shaded Fraunhofer lines. Close to the central absorption line there was also an unmistakable increase of luminosity (resembling the supposed emission lines in the solar spectrum); but this increase ought, without doubt, to be attributed to the most strongly curved rays being kept together by the tubular structure of the flame, and not to direct radiation from the flame. For, the electric light being intercepted, the emission-lines were scarcely visible in the dark field. And besides, as soon as the flame was disturbed by blowing upon it, or when it was partially covered by a diaphragm, the bright band, as well as the shading, became unsymmetrical with respect to the absorption line. Neither DOPPLER's principle, nor the influence of pressure on wave-length can here have played an appreciable part.

I moreover observed fringe-like maxima and minima in the shadings, but they showed irregular and so unsteady, that I could not think of measuring their distances. Nor can there be any question of photographing this peculiarity before means have been devised to keep a structure of sodium vapour, as described above, steady for a reasonable time. Such means are however being prepared.

Imperfect as our present experiment must be, it still serves to bear out the assertion, that numerous peculiarities of the solar spectrum may be explained from anomalous dispersion.

Physics. — *"On the emission and absorption by metals of rays of heat of great wave-lengths."* By H. A. LORENTZ.

§ 1. HAGEN and RUBENS have recently shown by their measurements of the reflecting power of metals [1]) that the behaviour of these bodies towards rays of great wave-lengths (larger than 8 μ) may be accounted for, if one applies to the propagation of electric vibrations the equations that hold for slowly varying currents, and which contain no other physical constant of the metal but its conductivity. It follows from this result that a theory which can give an adequate idea of the mechanism of a current of conduction will also suffice for the explanation of the absorption of the rays that have been used by these experimenters. A theory of this kind has been developed by RIECKE [2]) and DRUDE [3]). According to their views a metal contains an immense number of free electrons moving to and fro in much the same way as the molecules of a gas or as the ions in an electrolytic solution,

[1]) HAGEN and RUBENS, Berliner Sitzungsberichte, 1903, p. 269; Berichte d. deutschen phys. Gesellsch., 1903, p. 145.
[2]) RIECKE, Wied. Ann., Bd. 66, p. 353, 1898.
[3]) DRUDE, Drude's Ann., Bd. 1, p. 566, 1900.

the velocity of agitation increasing with the temperature. It is to be assumed that, in this "heat-motion", every electron travels along a straight line, until it strikes against a particle of the metal; the path will therefore be an irregular zigzag-line and, so long as there is no cause driving the electrons in a definite direction, an element of surface will be traversed by equal numbers of electrons, travelling to opposite sides. Things will be different if the metal is exposed to an electric force. The motion of the electrons will still be an irregular agitation; yet, motions in a definite direction will predominate, and this will show itself in our observations as an "electric current."

Now we may infer from the relation between absorption and emission that is required by Kirchhoff's law, that the mechanism by which the emission of a body is produced is the same as that to which it owes its absorbing power. It is therefore natural to expect that, if we confine ourselves to the case of great wave-lengths, we shall be able to explain the emission of a metal by means of the heat-motion of its free electrons, without recurring to the hypothesis of "vibrators" of some kind, producing waves of definite periods.

In the following pages this idea has been worked out. After having calculated the emissive power we shall find that its ratio to the absorbing power does not depend on the value of those quantities by which one metal differs from another. According to the law of Kirchhoff, the result may be considered as representing the ratio between the emission and the absorption for an arbitrarily chosen body, or as the emissive power of a perfectly black substance; it will be found to contain a certain constant quantity, whose physical meaning will appear from the theory.

§ 2. The ratio of which I have just spoken is intimately connected with another important physical quantity, viz. the density of the energy of radiation in a space enclosed by perfectly black walls, which are kept at a uniform absolute temperature T. If the electromagnetic motions of which the aether in such a space is the seat, are decomposed into rays travelling in all directions, and each of which has a definite wave-length, the energy per unit volume, in so far as it belongs to rays with wave-lengths between λ and $\lambda + d\lambda$, may be represented by

$$F(\lambda, T)\, d\lambda,$$

F being a function which many physicists have tried to determine. Boltzmann and Wien have shown by thermodynamical reasoning that the above expression may be written

$$\frac{1}{\lambda^5} f(\lambda T)\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f(\lambda T)$ is a function of the product λT. Afterwards PLANCK [1]) has found for (1) the form

$$\frac{8\pi c h}{\lambda^5} \cdot \frac{1}{e^{\frac{ch}{kT}} - 1}\, d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here c is the velocity of light in aether and h and k are universal constants.

In the theory of PLANCK every ponderable body is supposed to contain a great many electromagnetic vibrators, or, as PLANCK calls them, "resonators", each of which has its own period of free vibration, and which exchange energy with the aether as well as with the molecules or atoms of ponderable matter. The conditions of statistical equilibrium between the resonators and the aether may be thoroughly investigated by means of the equations of the electromagnetic field. As to the partition of energy between the vibrations of the resonators and the molecular motions in the body, PLANCK has not endeavoured to give an idea of the processes by which it takes place. He has used other modes of reasoning, of which I shall only mention one, which is to be found in his later papers on the subject and which consists in the determination of that distribution of energy that is to be considered as the most probable. I shall not here discuss the way in which the notion of probability is introduced in PLANCK's theory and which is not the only one that may be chosen. It will suffice to mention an assumption that is made about the quantities of energy that may be gained or lost by the resonators. These quantities are supposed to be made up of a certain number of *finite* portions, whose amount is fixed for every resonator; according to PLANCK, the energy that is stored up in a resonator cannot increase or diminish by gradual changes, but only by whole "units of energy", as we may call the portions we have just spoken of. Besides, PLANCK has found it necessary to ascribe to these units a magnitude depending on the frequency n of the free vibrations of the resonator, the magnitude being represented by $\frac{hn}{2\pi}$.

As to the constant k, it has a very simple physical meaning; $\frac{3}{2} kT$ is the mean kinetic energy of the molecule of a gas at the temperature T.

1) PLANCK, Drude's Ann., Bd. 1, p. 69, 1900; Bd. 4, p.p. 553 and 564, 1901.

It appears from the above remarks that the hypothesis regarding the finite "units of energy", which has led to the introduction of the constant h, is an essential part of the theory; also that the question as to the mechanism by which the heat of a body produces electromagnetic vibrations in the aether is still left open. Nevertheless, the results of PLANCK are most remarkable. His formula represents very exactly the energy of the radiations for all values of the wave-lengths, whereas the following considerations are from the outset confined to long waves. We may at best expect to deduce from them the form which the function in (1) takes for this extreme case.

§ 3. Since, if we trust to KIRCHHOFF's law, the ratio between the emission and the absorption must be regarded as independent of the dimensions and the position of the body considered, we may simplify the problem by an appropriate choice of circumstances. I shall therefore consider a plate with parallel plane surfaces and I shall suppose its thickness Δ to be so small that the absorption may be reckoned proportional to it and that the energy emitted by the posterior layers may be supposed to pass through the plate without any sensible absorption. I shall also confine myself to the absorption of perpendicularly incident rays and to the emission in directions making infinitely small angles with the normal.

Let σ be the conductivity of the metal, i. e. the constant ratio between the electric current and the electric force, these latter quantities being expressed in the modified electrostatic units I have lately introduced. [1]) Then the absorbing power of the plate, the coefficient by which we must multiply the energy of normal incident rays, in order to get the absorbed energy, is given by [2])

$$A = \frac{\sigma}{c} \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here we shall substitute for σ the value furnished by DRUDE's theory. Let the metal contain different kinds of free electrons, which we may distinguish as the 1st, the 2nd, the 3rd kind, etc., and let us suppose that all electrons of one and the same kind have equal charges, equal velocities of heat-motion, or, as we may say, "molecular" velocities, and travel over paths of equal mean length between two successive encounters with particles of the metal.

We shall write e_1, e_2, ... for the charges of the different kinds of electrons, u_1, u_2, ... for the mean molecular velocities, l_1, l_2, ...

[1]) LORENTZ, Proceedings Acad. of Science, Amsterdam, Vol. 11, p. 608, 1903.
[2]) See § 12 below. In electromagnetic units the formula becomes

$$A = 4\pi c \sigma \Delta.$$

for the mean lengths of the free paths, N_1, N_2, ... for the number of electrons of the several kinds, contained in unit of volume. We shall finally suppose, as DRUDE has done, that for every kind of electrons, the mean kinetic energy of one of these particles is equal to that of a molecule of a gas at the same temperature; we may represent it by αT, if T is the absolute temperature, and α a constant.

In these notations DRUDE's value is [1])

$$\sigma = \frac{1}{4\alpha T}(e_1^2 N_1 l_1 u_1 + e_2^2 N_2 l_2 u_2 + \ldots), \quad . \quad . \quad . \quad (4)$$

so that (3) becomes

$$A = \frac{1}{4\alpha c T}(e_1^2 N_1 l_1 u_1 + e_2^2 N_2 l_2 u_2 + \ldots)\,\Delta. \quad . \quad . \quad (5)$$

It is to be remarked that the formula (4) has been obtained in the supposition that the electric force remains constant, or at least that it keeps its direction and magnitude during an interval of time in which an electron has undergone a large number of collisions against particles of the metal. The results of HAGEN and RUBENS are therefore favorable to the view that even the period of vibration of the rays is very large in comparison with the time between two succeeding impacts. Part of the following calculations are based on this assumption.

§ 4. We have now to examine the emission by the plate. It follows from the fundamental equations of the theory of electrons, that every *change*, whether in direction or in magnitude, of the velocity of an electron produces an electromagnetic disturbance travelling outwards in the surrounding aether. Hence, it will be at the instants of the collisions that the electrons become centres of radiation. We shall calculate the amount of energy, radiated in this way, in so far as it is emitted across a definite part ω of the front surface of the plate; this part of the emission is due to the electrons contained in a volume $\omega\Delta$ of the metal.

Let O be a point within the area ω, OP the normal in this point, drawn towards the side of the aether, and P a point on this line, at a distance r from O, which is very large in comparison with the dimensions of ω. In this point P we place an element of surface ω', perpendicular to OP; our problem will be to calculate the energy radiated across this element. I choose O as origin of coordinates and OP as the axis of z. The components of the velocity of an electron will be denoted by u_x, u_y, u_z.

---

1) DRUDE, l. c., p. 576. This formula does not change by the introduction of our new units.

Now, if an electron with charge e, is in O at the time t, and has at that instant the accelerations $\frac{du_x}{dt}$, $\frac{du_y}{dt}$, $\frac{du_z}{dt}$, it will produce at the point P, at the time $t+\frac{r}{c}$, a dielectric displacement, whose components are [1])

$$-\frac{\mathrm{e}}{4\pi c^2 r}\frac{du_x}{dt}, \qquad -\frac{\mathrm{e}}{4\pi c^2 r}\frac{du_y}{dt}, \quad 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

On account of the great length of OP, these expressions may also be applied to an electron situated, not in O but in any other point of the part of the plate corresponding to the area ω. The whole dielectric displacement in P in the direction of x (it is only this component that will be considered in the next paragraphs) at the time $t+\frac{r}{c}$ will therefore be

$$\mathfrak{d}_x = -\frac{1}{4\pi c^2 r}\Sigma\, \mathrm{e}\frac{du_x}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

if the sum is extended to all electrons present in the volume $\omega\Delta$ at the time t.

There will also be a magnetic force of the same numerical value, and by POYNTING's theorem a flow of energy across the element ω', in the direction from the plate towards P. The amount of this flow per unit of time is given by

$$c\,\mathfrak{d}_x^2 \,.\, \omega' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

§ 5. It will be necessary for our purpose to decompose the whole emission into rays of different wave-lengths and to examine the part of (8) corresponding to the rays that have their wave-lengths within certain limits. This may be done by means of FOURIER's series.

Let us consider a *very long* time, extending from $t=0$ to $t=\vartheta$. During this interval the value of $\mathfrak{d}_x$ at the point P will continually change in a very irregular way; it may however in every case be expanded in the series

$$\mathfrak{d}_x = \sum_{m=1}^{m=\infty} a_m \sin\frac{m\pi t}{\vartheta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whose coefficients are given by

$$a_m = \frac{2}{\vartheta}\int_0^{\vartheta} \sin\frac{m\pi t}{\vartheta}\,\mathfrak{d}_x\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[1]) The proof of this will be found in one of the next parts of my "Contributions to the theory of electrons."

Now, if the plate is kept at a constant temperature, the radiation will also be stationary and $\mathfrak{d}_x^2$ may be replaced by its mean value

$$\overline{\mathfrak{d}_x^2} = \frac{1}{\vartheta}\int_0^{\vartheta} \mathfrak{d}_x^2\, dt$$

during the time ϑ. Substituting the value (9), we get integrals of two different kinds, some containing the square of a sine, and others the product of two sines. The integrals of the second kind will disappear, and

$$\int_0^{\vartheta} \sin^2 \frac{m\pi t}{\vartheta}\, dt = \frac{1}{2}\,\vartheta,$$

so that

$$\overline{\mathfrak{d}_x^2} = \frac{1}{2}\sum_{m=1}^{m=\infty} a_m^2. \quad \ldots \ldots \ldots \quad (11)$$

As to the frequency of the terms in (9), it is given by

$$n = \frac{m\pi}{\vartheta}; \quad \ldots \ldots \ldots \quad (12)$$

it will therefore increase by equal differences $\frac{\pi}{\vartheta}$, if we give to m its successive values.

By choosing for ϑ a value sufficiently large, we may make this step $\frac{\pi}{\vartheta}$ as small as we like, so that ultimately, even between two values of the frequency n and $n + dn$, which are in a physical sense infinitely near each other, there will be a certain number of values of (12) and of corresponding terms in the series (11). The number of these terms will be $\frac{\vartheta}{\pi}\, dn$, hence, if we suppose a_m, or

$$a_m = \frac{2}{\vartheta}\int_0^{\vartheta} \sin nt \,.\, \mathfrak{d}_x\, dt. \quad \ldots \ldots \ldots \quad (13)$$

to have the same value for each term of this group, the corresponding part of (11) will be

$$\frac{\vartheta}{2\pi}\, a_m^2\, dn.$$

Substituting this for $\mathfrak{d}_x^2$ in (8), we get for the radiation across ω', due to the rays with frequencies between n and $n + dn$,

$$\frac{c\,\vartheta}{2\pi}\,\omega'\, a_m^2\, dn. \quad \ldots \ldots \ldots \quad (14)$$

§ 6. We have now to calculate the coefficient a_m by means of (13). After having substituted in the integral the value (7), we may still take for its limits 0 and ϑ, provided we reckon the time from an instant, preceding by the interval $\frac{r}{c}$ the moment from which it has been reckoned till now. Thus:

$$a_m = -\frac{1}{2\pi c^2 \vartheta r} \Sigma \left[\mathrm{e} \int_0^{\vartheta} \sin nt \, . \, \frac{du_x}{dt} \, dt \right],$$

or, after integration by parts, since $\sin nt$ vanishes at the limits,

$$a_m = \frac{n}{2\,\pi\, c^2\, \vartheta\, r} \Sigma \left[\mathrm{e} \int_0^{\vartheta} \cos nt \, . \, u_x \, dt \right]. \quad . \quad . \quad . \quad . \quad (15)$$

The sum in these expressions relates to all the electrons in the part $\omega\Delta$ of the plate and it is by reason of the immense number of these particles that a definite value may be assigned to a^2_m.

We shall begin by determining a^2_m and the amount of the radiation in the supposition that there are only free electrons of one kind (§ 3). We shall write $q = N\omega\Delta$ for their number, e for the charge of each of them, and we shall further simplify the problem by supposing that the molecular velocity u, the same for all the electrons, is not altered by the collisions and that all the paths between two successive impacts have exactly the same length l. Then, the time

$$\tau = \frac{l}{u}$$

will also have a definite length.

§ 7. Let $t_1, t_2, t_3, \ldots$ be a series of instants, between 0 and ϑ, at intervals τ from each other. Then it is clear that, if we fix our attention on the positions of a single electron at these instants, we shall have one point on each of the sides of the zigzag-line described by this particle.

Now we may in the first place determine the integral in (15) for the lapse of time during which an electron travels over the side of the zigzag-line on which it is found at the time t_k. As the length τ of this interval is much shorter than the period $\frac{2\pi}{n}$ of the factor $\cos nt$, we may write for the integral

$$\cos nt_k \, . \, \tau u_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is clear that we shall obtain the sum in (15), for the q electrons, if, after having multiplied (16) by e, we perform the two summations, indicated in the formula

$$a_m = \frac{n e \tau}{2 \pi c^2 \vartheta r} \sum_k [\cos n t_k \Sigma u_x]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We have in the first place to take the sum of all the values of u_x for the system of electrons, at a particular instant t_k, and then to add together all the results obtained in this way for the instants t_1, t_2, etc.

§ 8. If we wish to find Σu_x for a given time, we must keep in mind that the velocities u of the electrons have at that instant very different directions. We may represent all these velocities by vectors drawn from a fixed point C. The ends D of all these vectors will lie on a sphere with radius u, and if we let fall from each of these points a perpendicular $D D'$ on the diameter of this sphere that is parallel to OX, the distances of the projections from C will give the values of u_x. The sum of all these values may therefore be represented by

$$\Sigma u_x = q \xi,$$

if ξ is the positive or negative distance at which the centre of gravity of the points D', considered as equal to each other, is situated from the centre C.

Of course, on account of the large number of the points, this distance will be very much smaller than the radius u, and, if we repeat the construction of the diagram of velocities for each of the instants $t_1, t_2 \ldots$, the small value that is found for ξ will be positive in one case and negative in another. It is to be remarked in this respect that there is no connexion at all between the values of ξ, which we shall find for two succeeding instants in the series $t_1, t_2 \ldots$ Indeed, between any two such instants, every electron will have undergone a collision, and it may safely be assumed that, whatever be the direction of motion of an electron before the impact, all directions will be equally probable after the impact [1]).

Now, in order to determine a^2_m, we have to take the square of the sum denoted by $\sum_k$ in the formula (17). This square consists of terms of two kinds, some having the form

$$\cos^2 n t_k [\Sigma u_x]^2_{t_k} = q^2 \cos^2 n t_k \xi^2_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

1) This is easily shown, as has been done by MAXWELL in his first paper on the kinetic theory of gases, if both the electrons and the particles of the metal are supposed to be perfectly elastic spheres.

and others the form

$$2 \cos n t_k \cos n t_{k'} [\Sigma u_x]_{t_k} [\Sigma u_x]_{t_{k'}} = 2 q^2 \cos n t_k \cos n t_{k'} \xi_k \xi_{k'} . \quad (19)$$

As has already been said, the time ϑ contains a very large number of periods $\frac{2\pi}{n}$. A certain value of $\cos nt$, once occurring in the series $\cos n t_1$, $\cos n t_2$, $\cos n t_3, \ldots$ may therefore be supposed to repeat itself many times. Also, one and the same value of the product $\cos n t_k \cos n t_{k'}$ may be said to occur for many different values of k and k'. Such a product will therefore have to be multiplied by very different expressions of the form $\xi_k \xi_{k'}$, and, since the different values of ξ are mutually independent, the number of cases in which ξ_k and $\xi_{k'}$ have opposite signs will be equal to that in which they have the same sign. It appears in this way that the terms (19) will cancel each other in the sum. It is only the terms of the form (18) that remain, and we shall have

$$a_m^2 = \frac{n^2 e^2 \tau^2 q^2}{4 \pi^2 c^4 \vartheta^2 r^2} \sum_k [\cos^2 n t_k \cdot \xi_k^2] \quad . \quad . \quad . \quad . \quad . \quad (20)$$

§ 9. Here we may begin by taking together those terms in which $\cos n t_k$ has one and the same value. Let the number of these be Q. Then, we have to repeat Q times the construction of the diagram of velocities, and it may be asked in how many of these Q cases ξ will lie between given limits ξ and $\xi + d\xi$, or, what amounts to the same thing, what is the probability for ξ falling between these limits.

This question may be reduced to a simpler problem. A series of planes, perpendicular to OX and at equal distances from one another, will divide the spherical surface into equal parts. Therefore, instead of distributing the points D on the surface in an irregular, arbitrarily chosen manner, we may as well immediately distribute the points D' at random over the diameter, without giving any preference to one part of the line over another. The probability in question is thus found to be [1])

$$P d\xi = \frac{1}{u} \sqrt{\frac{3q}{2\pi}} \, e^{-\frac{3q}{2u^2}\xi^2} \, d\xi . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, among the Q terms in the sum, occurring in (20), for which the factor $\cos^2 n t_k$ has equal values, there will be $QPd\xi$ terms, which may be said to have the same ξ_k. Together, they will contribute to the sum the amount

[1]) See §§ 13—15.

$$\cos^2 nt_k \,.\, Q P \xi^2 \, d\xi$$

and the total sum of all the Q terms is got from this by an integration which we may extend from $\xi = -\infty$ to $\xi = +\infty$. Consequently, the sum of those Q terms will not be altered, if, in each of them, we replace $\xi^2{}_k$ by

$$\overline{\xi^2} = \int_{-\infty}^{+\infty} P \xi^2 \, d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This expression being the same whatever be the particular value of $\cos^2 nt_k$, the sum in (20) at once becomes

$$\overline{\xi^2} \sum_k [\cos^2 nt_k] . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Again, since the instants t_1, t_2, ... are uniformly distributed at distances that are very small parts of the period $\frac{2\pi}{n}$, the sum will remain the same, if in every term we write $\frac{1}{2}$ instead of $\cos^2 nt_k$. The number of terms being $\frac{\vartheta}{\tau}$, we find for (23)

$$\frac{\vartheta}{2\tau} \overline{\xi^2}$$

and for (20)

$$a^2{}_m = \frac{n^2 e^2 \tau q^2}{8\pi^2 c^4 \vartheta r^2} \overline{\xi^2}.$$

We have by (21) and (22)

$$\overline{\xi^2} = \frac{u^2}{3q};$$

hence, replacing τ by $\frac{l}{u}$, we find

$$a^2{}_m = \frac{n^2 e^2 q l u}{24\pi^2 c^4 \vartheta r^2} = \frac{n^2 e^2 N l u \Delta}{24\pi^2 c^4 \vartheta r^2} \, \omega$$

and for the emission (14), in so far as it is due to the one kind of electrons that has been considered

$$\frac{n^2 e^2 N l u \Delta}{48\pi^3 c^3 r^2} \, \omega \, \omega' \, dn.$$

This value must still be multiplied by 2 because we may apply to the second of the components (6) the same reasoning as to the first component, and the total radiation from the plate may obviously be considered as the sum of all the values corresponding to the different kinds of electrons. The final result is therefore [1])

$$\frac{n^2}{24\pi^3 c^3 r^2}(e_1^2 N_1 l_1 u_1 + e_2^2 N_2 l_2 u_2 + \ldots)\,\Delta\,\omega\,\omega'\,dn. \quad . \quad (24)$$

§ 10. If now we divide (24) by (5), all quantities N, e, u and l, by which one metal differs from another, disappear. This is what might be expected according to KIRCHHOFF's law and the result

$$\frac{an^2 T}{6\pi^3 c^2 r^2}\,\omega\,\omega'\,dn$$

may be taken to express the emission by a perfectly black body under the circumstances we have supposed. It represents the amount of energy which, in the case of such a body, is transmitted per unit of time across an element ω', in the rays whose frequency lies between n and $n + dn$ and whose directions deviate infinitely little from the normal to the element, being contained within a solid angle $\frac{\omega}{r^2}$. Multiplying by $\frac{4\pi r^2}{c\omega\omega'}$, we are led to the following expression for the density of energy of which I have spoken in § 2:

$$\frac{2an^2 T}{3\pi^2 c^3}\,dn. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Taking for the group of rays those whose wave-lengths are included between λ and $\lambda + d\lambda$, we get for the corresponding energy per unit volume

$$\frac{16}{3}\,\frac{\pi a T}{\lambda^4}\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

---

[1]) It is easy to free ourselves from the hypothesis that for all electrons of one kind there is a single length of path l and a single molecular velocity u. Indeed, the motion of an electron along one of the small straight lines l, which it describes between the instants 0 and ϑ, will furnish for the sum in (15) a quantity

$$e \cos nt \,.\, u_x\, \tau,$$

if u is the velocity for the particular line l we wish to consider, and τ the time required for the motion along it.

Now, among all these rectilinear motions between two successive encounters, of one kind of electrons, we may select those for which u and l have certain definite values and we may begin by calculating the coefficient a_m and the emission, in so far as they depend on the part of (15) which corresponds to these particular motions; in doing so, we may use the method shown in §§ 7—9. The total emission may be regarded as the sum of all the partial values (with different l's and different u's) thus obtained, and after all the expression (24) will still hold, provided we understand by $l_1, l_2 \ldots$ certain mean lengths of path and by $u_1, u_2 \ldots$ certain mean molecular velocities. We need not however enter into these details, because the conductivity and the coefficient of absorption have not been calculated with a corresponding degree of accuracy.

This is found from (25) by using the relation $n = \frac{2\pi c}{\lambda}$.

§ 11. The result of the preceding calculations not only conforms to the law of KIRCHHOFF; it has also a form agreeing with those of BOLTZMANN and WIEN. Indeed, the expression (26) follows from (1), if we put

$$f(\lambda T) = \frac{16}{3} \pi \alpha \,.\, \lambda T.$$

Our last task will be to evaluate the constant α by applying the formula (26) to experimental determinations of the radiation of black bodies, and to compare the result with what has been inferred about the same constant from other classes of phenomena. Combining the measurements of LUMMER and PRINGSHEIM [1]), who have gone far into the infra-red, with the absolute amount of the radiation as determined by KURLBAUM [2]), I find

$$\alpha = 1{,}6 \,.\, 10^{-16} \frac{erg}{degree}.$$

On the other hand, we get, starting from VAN DER WAALS' evaluation of the mass of an atom of hydrogen,

$$\alpha = 1{,}2 \,.\, 10^{-16}.$$

A comparison of my formula with that of PLANCK is also interesting. For very large values of the product λT, the denominator in (2) becomes $\frac{ch}{k\lambda T}$, and the expression itself $\frac{8\pi k T}{\lambda^4} d\lambda$. This agrees with (26), if $\alpha = \frac{3}{2} k$.

Now the mean kinetic energy of a molecule of a gas would be $\frac{3}{2} kT$ according to PLANCK and has been represented in what precedes by αT. There appears therefore to be a full agreement between the two theories in the case of long waves, certainly a remarkable conclusion, as the fundamental assumptions are widely different.

On the absorption by a thin metallic plate.

§ 12. Take the origin of coordinates in the front surface, the axis of z towards the metal, and let there be free aether on both sides.

Writing $\mathfrak{E}$ for the electric force, $\mathfrak{J}$ for the current of conduction,

1) LUMMER and PRINGSHEIM, Verhandl. d. deutschen phys. Gesellsch., 1900, p. 163.
2) KURLBAUM, Wied. Ann., Bd. 65, p. 754, 1898.

$\mathfrak{H}$ for the magnetic force and putting the magnetic permeability $= 1$, we have for the metal

$$rot\ \mathfrak{H} = \frac{1}{c}\mathfrak{J}, \qquad rot\ \mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{H}}, \qquad \mathfrak{J} = \sigma\mathfrak{E}.$$

It is found by these equations that in electromagnetic waves travelling in the direction of the positive z, $\mathfrak{E}$ and $\mathfrak{H}$ can have the directions of OX and OY, and values equal to the real parts of the complex quantities

$$\mathfrak{E}_x = a\,e^{int-\alpha(1+i)z}, \qquad \mathfrak{H}_y = \varkappa\, a\, e^{int-\alpha(1+i)z} \quad . \quad . \quad (27)$$

a being the amplitude of the electric force, and the constants α and $\varkappa$ being given by

$$\alpha = \frac{1}{c}\sqrt{\frac{1}{2}n\sigma}, \qquad \varkappa = (1-i)\sqrt{\frac{\sigma}{2n}}.$$

Similarly, waves travelling in the opposite direction may be represented by

$$\mathfrak{E}_x = a\,e^{int+\alpha(1+i)z}, \qquad \mathfrak{H}_y = -\varkappa\, a\, e^{int+\alpha(1+i)z} \quad . \quad . \quad (28)$$

For the aether the corresponding formulae are somewhat simpler; in the first case

$$\mathfrak{E}_x = a\,e^{int-i\frac{n}{c}z}, \qquad \mathfrak{H}_y = a\,e^{int-i\frac{n}{c}z} \quad . \quad . \quad . \quad . \quad (29)$$

and in the second

$$\mathfrak{E}_x = a\,e^{int+i\frac{n}{c}z}, \qquad \mathfrak{H}_y = -a\,e^{int+i\frac{n}{c}z} \quad . \quad . \quad . \quad . \quad (30)$$

Now, if rays fall perpendicularly on the front surface of the plate, we may unite all the systems of waves arising from the repeated reflexions into the following parts: 1st. a reflected system in the aether, 2nd. transmitted waves in the aether behind the plate, 3rd. waves in the plate, travelling towards the back surface and 4th. rays in the metal, going in the opposite direction. Representing the incident rays and the motions mentioned under these four heads by the equations (29), (30), (29), (27), (28), with the values a_1, a_2, a_3, a_4, a_5 of the amplitude, we have, in virtue of the conditions at the two surfaces (continuity of $\mathfrak{E}_x$ and $\mathfrak{H}_y$)

$$a_1 + a_2 = a_4 + a_5,$$
$$a_1 - a_2 = \varkappa(a_4 - a_5),$$

$$a_4\,e^{-s} + a_5\,e^{+s} = a_3\,e^{-i\frac{n}{c}\Delta},$$

$$\varkappa\, a_4\,e^{-s} - \varkappa\, a_5\,e^{+s} = a_3\,e^{-i\frac{n}{c}\Delta}.$$

In these formulae, Δ is the thickness of the plate, and

$$\alpha (1 + i) \Delta = s \quad \ldots \ldots \quad (31)$$

The solution, in so far as it is necessary to our purpose, is

$$a_2 = \frac{(\varkappa^2 - 1)(e^{-s} - e^{+s})}{(\varkappa + 1)^2 e^{+s} - (\varkappa - 1)^2 e^{-s}} a_1,$$

$$a_3 = \frac{4 \varkappa}{(\varkappa + 1)^2 e^{+s} - (\varkappa - 1)^2 e^{-s}} e^{i \frac{n}{c} \Delta} a_1.$$

In these expressions Δ and consequently s are now to be supposed infinitely small. Replacing e^{-s} and e^{+s} by $1 - s$ and $1 + s$, one finds

$$a_2 = -\frac{1}{2}\left(\varkappa - \frac{1}{\varkappa}\right) s a_1,$$

$$a_3 = \left[1 - \frac{1}{2}\left(\varkappa + \frac{1}{\varkappa}\right) s\right] e^{i \frac{n}{c} \Delta} a_1$$

The first of these equations shows that the amplitude of the rays reflected by the thin plate is infinitely small, so that we may neglect their energy as a quantity of the second order.

As to the transmitted rays, the amount of energy propagated in them will be equal to the product of the incident energy by the square of the modulus of the complex expression

$$\left[1 - \frac{1}{2}\left(\varkappa + \frac{1}{\varkappa}\right) s\right].$$

This square is

$$1 - \frac{\sigma}{c} \Delta,$$

whence we deduce for the coefficient of absorption

$$A = \frac{\sigma}{c} \Delta.$$

On the probability with which one may expect that the centre of gravity of a large number of points distributed at random on a limited straight line will lie within given limits.

§ 13. Divide the line into a large number p of equal parts, and call these, beginning at the end A of the line, the 1st, the 2nd, the 3rd part, etc. Denote by q the number of points and let q be very much larger than p.

We shall imagine the points to be placed on the line one after another, in such a way that, whatever be the position of the points already distributed, a new point may as well fall on one part of the line as on the other. The result will be a certain distribution of the whole number, entirely determined by chance. Let us conceive this operation to be very often repeated, say Q times, and let us calculate in how many of these Q cases, a desired distribution of the points over the p parts will occur. Dividing by Q we shall have the probability of the distribution.

The probability that there will be $a, b, \ldots m$ points on the 1st, 2nd, ... pth part of the line $(a + b + \ldots + m = q)$, is given by

$$P = \left(\frac{1}{p}\right)^q \frac{q!}{a!\,b!\ldots m!}.$$

In the case of a very large value $\frac{q}{p}$, this probability becomes extremely small, as soon as one of the numbers $a, b, \ldots m$ is far below $\frac{q}{p}$. Neglecting these small probabilities, we shall confine ourselves to those cases, in which each of the numbers $a, b, \ldots . m$ is very large. Then, by the well known formula of Stirling,

$$a! = \sqrt{2a\pi}\left(\frac{a}{e}\right)^a, \text{ etc.}$$

and, if we put

$$\frac{a}{q} = a', \frac{b}{q} = b', \ldots \frac{m}{q} = m',$$

we shall find

$$\begin{aligned} \log P = &- \tfrac{1}{2}(p-1)\log(2\pi q) - q \log p - \\ &- [(a'q + \tfrac{1}{2})\log a' + \ldots + (m'q + \tfrac{1}{2})\log m'] \quad . \quad . \quad . \quad . \quad (32) \end{aligned}$$

It is to be remarked that the numbers $a, b, \ldots m$ can only increase or diminish by whole units. The numbers $a', b' \ldots m'$ can change by steps equal to $\frac{1}{q}$; this may be made so small that they may be considered as continuously variable.

§ 14. We shall in the first place determine the values of $a', b', \ldots m'$ for which the probability P becomes a maximum. We have

$$d \log P = -\left[\left(q + \frac{1}{2a'} + q \log a'\right)da' + \ldots + \left(q + \frac{1}{2m'} + q \log m'\right)dm'\right],$$

with the condition

$$da' + \ldots + dm' = 0,$$

which is a consequence of

$$a' + \ldots . + m' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The maximum will therefore be reached if

$$a' = b' = \dots = m' = \frac{1}{p},$$

so that the uniform distribution will be the most probable.

We shall next consider the probability for a distribution differing a little from the most probable one. Let us put

$$a' = \frac{1}{p} + \alpha \quad , \quad b' = \frac{1}{p} + \beta \quad , \quad \dots \; m' = \frac{1}{p} + \mu \quad . \quad (34)$$

and let us suppose the numbers $\alpha, \beta \dots \mu$, to be so small in comparison with $\frac{1}{p}$, that in the expansion of the quantities in (32) in ascending powers of $\alpha, \beta \dots \mu$, we may neglect all powers surpassing the second. We have for instance

$$\left(a'q + \frac{1}{2}\right) \log a' = -\left(\frac{q}{p} + \frac{1}{2}\right) \log p + \left(q + \frac{1}{2} p - q \log p\right) \alpha + \frac{1}{2} p \left(q - \frac{1}{2} p\right) \alpha^2,$$

where, in the last term, we may omit the term $\frac{1}{2} p$, because it is much smaller than q. If we put

$$-\frac{1}{2} (p-1) \log (2 \pi q) + \frac{1}{2} p \log p = \log P_m$$

and keep in mind that, in virtue of (33),

$$\alpha + \beta + \dots + \mu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

the equation (32) becomes

$$\log P = \log P_m - \frac{1}{2} p\, q\, (\alpha^2 + \beta^2 + \dots + \mu^2),$$

$$P = P_m \, e^{-\frac{1}{2} p\, q\, (\alpha^2 + \beta^2 + \dots + \mu^2)}.$$

It is seen from this that P_m is the maximum of the probability, with which we shall have to do, if $\alpha = \beta = \dots = \mu = 0$. The equation shows also that, conformly to what has been said above, the probability will only be comparable to P_m so long as $\alpha, \beta \dots \mu$ are far below $\frac{1}{p}$. Indeed, if one of these numbers had this last value, P_m would be multiplied by

$$e^{-\frac{q}{2p}},$$

which, by our assumptions, is extremely small.

§ 15. Let $2u$ be the length of the line, x the distance along the line, reckoned from the end A, and let us take $\frac{u}{p}$ for the value of this coordinate for all points situated on the first part of the line, $3\frac{u}{p}$ for all points of the second part, and so on. Then, in the distribution that is characterized by $a', b', \ldots. m'$, the coordinate of the centre of gravity of the q points will be

$$[a' + 3b' + 5c' + \ldots. + (2p-1)m']\frac{u}{p},$$

or, by (34),

$$u + [\alpha + 3\beta + 5\gamma + \ldots. + (2p-1)\mu]\frac{u}{p}.$$

The positive or negative value of

$$\xi = [\alpha + 3\beta + 5\gamma + \ldots. + (2p-1)\mu]\frac{u}{p}. \quad . \quad . \quad (36)$$

is thus seen to represent the distance between the middle point of the line and the centre of gravity. We have to calculate the probability for this distance lying between ξ and $\xi + d\xi$.

The problem is easily solved by means of a change of variables. Instead of the quantities $\alpha, \beta, \ldots. \mu$, which serve to define a mode of distribution, we shall introduce new ones $\alpha', \beta', \ldots. \mu'$, the substitution being linear and orthogonal.

Let us take for the first of the new variables

$$\alpha' = \frac{1}{\sqrt{p}}\alpha + \frac{1}{\sqrt{p}}\beta + \ldots + \frac{1}{\sqrt{p}}\mu, \quad . \quad . \quad . \quad . \quad (37)$$

and for the second

$$\beta' = -\frac{p-1}{\varkappa}\alpha - \frac{p-3}{\varkappa}\beta - \ldots + \frac{p-1}{\varkappa}\mu, \quad . \quad . \quad . \quad (38)$$

where the numerators form an arithmetical progression, whereas $\varkappa$ means the positive square root of the sum of the squares of the numerators. These expressions (37) and (38) may really be adopted, because the peculiar conditions for an orthogonal substitution are satisfied: in both expressions the sum of the squares of the coefficients is 1, and we get 0 if we add together the coefficients of (37) after having multiplied them by the corresponding coefficients in (38). As to the coefficients in the expressions for $\gamma', \ldots \mu'$, we may choose them as we like, provided the whole substitution remain orthogonal.

The reason for the above choice of α' and β' will be clear; the condition (35) simplifies to

$$\alpha' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and, in virtue of (35), the value (36) will be equal to

$$\xi = \frac{\varkappa u}{p}\beta' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in all cases with which we are concerned.

Now, the modes of distribution for which the value of ξ lies between ξ and $\xi + d\xi$ are those for which β' lies between β' and $\beta' + d\beta'$, if

$$d\beta' = \frac{p}{\varkappa u}\, d\xi. \quad \quad (41)$$

Since $\alpha' = 0$, every mode of distribution may be defined by the values of $\beta' \ldots \mu'$, these quantities being, like $\alpha, \beta, \ldots \mu$, capable of very small variations.

We can therefore select, among all the modes of distribution, those for which $\beta' \ldots \mu'$ lie between β' and $\beta' + d\beta'$, γ' and $\gamma' + d\gamma'$, etc. The number of these may be represented by

$$h\, d\beta' \ldots d\mu', \quad \quad (42)$$

where h is a coefficient whose value need not be specified. It suffices to know that it is independent of the values chosen for $\beta' \ldots \mu'$. This is a consequence of the linear form of the relations between these variables and $a, b, \ldots m$.

As the just mentioned modes of distribution, whose number is given by (42), differ infinitely little from one another, the probability P may be taken to be the same for each of them. Hence, the probability for the occurrence of one of these modes, no matter which, must be

$$h\, P\, d\beta' \ldots d\mu' \quad \quad (43)$$

From this we may pass to the probability for β' lying between β' and $\beta' + d\beta'$, whatever be the values of $\gamma' \ldots \mu'$; we have only to integrate with respect to these last variables. Now using the fundamental property of an orthogonal substitution

$$\alpha^2 + \beta^2 + \ldots + \mu^2 = \alpha'^2 + \beta'^2 + \ldots + \mu'^2,$$

and attending to (39), we write for (43)

$$h\, P_m\, e^{-\frac{1}{2} p q (\beta'^2 + \ldots + \mu'^2)}\, d\beta' \ldots d\mu'.$$

If we integrate this expression from $-\infty$ to $+\infty$, as may be done for obvious reasons, denoting by k a coefficient that does not depend on β', we find for the probability in question

$$k\, e^{-\frac{1}{2} p q \beta'^2}\, d\beta'.$$

On account of (40) and (41) this is equal to

$$k'\, e^{-\frac{p^3 q}{2\varkappa^2 u^2}\xi^2}\, d\xi, \quad \quad (44)$$

k' being a new constant.

It remains to introduce the value of $\varkappa^2$. According to the definition of this quantity, it is $\frac{1}{3}p(p^2-1)$, instead of which we may take $\frac{1}{3}p^3$, because p is a very large number. In this way (44) changes into

$$k' e^{-\frac{3q}{2u^2}\xi^2} d\xi. \quad \ldots \ldots \ldots \quad (45)$$

We may finally determine the coefficient k' by remarking that (45), integrated from $-\infty$ to $+\infty$, must necessarily give 1. This requires that

$$k' = \frac{1}{u}\sqrt{\frac{3q}{2\pi}},$$

so that our result becomes

$$\frac{1}{u}\sqrt{\frac{3q}{2\pi}}\, e^{-\frac{3q}{2u^2}\xi^2}\, d\xi.$$

Microbiology. — "*The decomposition of cellulose by aërobic micro-organisms.*" By G. van Iterson Jr. (Communicated by Prof. M. W. Beijerinck).

(Communicated at the meeting March 28, 1903).

When we introduce into the soil or into natural waters substances consisting of cellulose, such as linen, cotton or paper, it will be seen, that the greater part comparatively soon disappears, whilst the cellulose derived from the continually falling dead leaves and other parts of plants is also soon destroyed under natural conditions either totally or partly. It is also a known fact that the layer of humus in the primeval forests has a limited thickness, the decrease being just compensated by the increase caused by the falling of the leaves. Investigation shows that the cellulose, although chemically so stable, is decomposed by micro-organisms. The observations by Mitscherlich [1]), Popoff [2]), van Tieghem [3]), Tappeiner [4]), van

[1]) Ueber die Zusammensetzung der Wand der Pflanzenzelle, Monatsber. d. Berl. Akad., 1850, p. 102.

[2]) Ueber Sumpfgasgährung, Archiv. f. ges. Physiol., 1875, Bd. 10, S. 113.

[3]) Sur le bacillus amylobacter et son rôle dans la putréfaction des tissus végétaux, C. R. t. 88, 1879, p. 88. — Identité du bacillus amylobacter et du vibrion butyrique de M. Pasteur, C. R. t. 89, 1879, p. 5.

[4]) Ueber Celluloseverdauung, Ber. d. d. ch. G. Bd. 15, 1882, S. 999. — Ueber Cellulosegährungen, Ber. d. d. ch. G. Bd. 16, 1883, S. 1734. — Ueber die Sumpfgasgahrung im Schlamme der Teiche, Sümpfe u. Kloaken, Ber. d. d. ch. G. Bd. 16, 1883, S. 1740.