Mathematics. — On "*Special cases of* MONGE's *Differential Equation*" by Prof. W. KAPTEYN.

When the differential equation of MONGE consists of three terms and has the form

$$s+\lambda t+\mu=0$$

we formerly found all the cases in which this equation has two intermediate integrals. A further investigation of the equation

$$r-\lambda^2 t+\mu=0$$

gives the following results.

I. When λ and μ are dependent only on p and q, this equation possesses two intermediate integrals only when

$$\lambda=\frac{a+2bp+cp^2}{f+2gq+hg^2}=\frac{P}{Q}$$

and

$$\mu=K\lambda(hQ-cP).$$

Here K represents an arbitrary constant, whilst the six constants a, b, c, f, g, h are bound only by the condition

$$b^2-ac=g^2-fh.$$

If we put $b^2-ac=\alpha^2$, the two intermediate integrals are the following

$$w^{g-\alpha}(w-1)^{2\alpha}e^{2\alpha hK[(g-b)x-cz]}=f\left(we^{2\alpha K(hx+cy)}\right),$$

where

$$w=\frac{b+cp-\alpha}{b+cp+\alpha}\,\frac{g+hq-\alpha}{g+hq+\alpha}$$

and where f represents an arbitrary function

$$w'^{-g-\alpha}(w'-1)^{2\alpha}e^{-2\alpha hK[(g+b)x+cz]}=f\left(w'e^{2\alpha K(hx-cy)}\right),$$

where

$$w_1 = \frac{b + cp - \alpha}{b + cp + \alpha} \quad \frac{g + hq + \alpha}{g + hq - \alpha}$$

f having the same meaning as above.

II. When λ and μ are only dependent on x, y, z, the equation possesses two intermediate integrals only when

$$\lambda = \frac{X'}{Y'},$$

$$\mu = \frac{X'^2}{4}\left[\frac{2\,X'\,X''' - 3\,X''^2}{X'^4} - \frac{2\,Y'\,Y''' - 3\,Y''^2}{Y'^4}\right]z + \psi(x, y),$$

where $\psi(x, y)$ represents an arbitrary function of x and y, whilst X denotes a function of x and Y a function of y, of which the derivatives are indicated by dashes.

In this case we find one of the intermediate integrals by eliminating y between

$$X + Y = \text{Const.} = C$$

and

$$\frac{dv}{dx} - \frac{3}{2}X'\left(\frac{X''}{X'^2} + \frac{Y''}{Y'^2}\right)v = -2\frac{X'}{Y'}\psi(x, y),$$

where

$$v = \left(\frac{\partial\lambda}{\partial x} - \lambda\frac{\partial\lambda}{\partial y}\right)z + 2\,\lambda\,(p + \lambda\,q).$$

If we solve the integral of this linear differential equation according to the arbitrary constant C' and if we replace this constant by an arbitrary function of C, the intermediate integral under research will be found if moreover we substitute $X + Y$ for C.

The second intermediate integral is determined in a similar way out of

$$X - Y = C$$

and

$$\frac{dv'}{dx} - \frac{3}{2}X'\left(\frac{X''}{X'^2} - \frac{Y''}{Y'^2}\right)v' = -2\frac{X'}{Y'}\psi(x, y),$$

where

$$v' = \left(\frac{\partial\lambda}{\partial x} + \lambda\frac{\partial\lambda}{\partial y}\right)z + 2\,\lambda\,(p - \lambda\,q).$$