Mathematics.. — *"A definite integral containing Bessel's functions"* by Prof. W. KAPTEYN.

If $I_m(t)$ and $I_n(t)$ are two Bessel's functions of the first kind and of orders m and n, then

$$\frac{d^2 I_m}{dt^2}+\frac{1}{t}\,\frac{d I_m}{dt}+\left(1-\frac{m^2}{t^2}\right) I_m=0$$

and

$$\frac{d^2 I_n}{dt^2}+\frac{1}{t}\,\frac{d I_n}{dt}+\left(1-\frac{n^2}{t^2}\right) I_n=0.$$

If we multiply the first equation by I_n and the last by I_m, we find by subtraction

$$I_n\frac{d^2 I_m}{dt^2}-I_m\frac{d^2 I_n}{dt}+\frac{1}{t}\left(I_n\frac{d I_m}{dt}-I_m\frac{d I_n}{dt}\right)=\frac{m^2-n^2}{t^2} I_m I_n.$$

By putting

$$I_n\frac{d I_m}{dt}-I_m\frac{d I_n}{dt}=U$$

we obtain

$$\frac{dU}{dt}+\frac{1}{t}\,U=\frac{m^2-n^2}{t} I_m I_n$$

or

$$\frac{d}{dt}(U\,t)=\frac{m^2-n^2}{t} I_m I_n\,;$$

and after integration between 0 and ∞

$$(U\,t)_0^\infty=(m^2-n^2)\int_0^\infty\frac{I_m I_n}{t}\,dt.$$

Now for $t=\infty$ we have

$$I_m = \sqrt{\frac{2}{\pi}}\frac{1}{t^{\frac{1}{2}}}\cos(t-\alpha),\ \alpha = \frac{2m+1}{4}\pi$$

$$I_n = \sqrt{\frac{2}{\pi}}\frac{1}{t^{\frac{1}{2}}}\cos(t-\beta),\ \beta = \frac{2n+1}{4}\pi$$

$$\frac{dI_m}{dt} = -\sqrt{\frac{2}{\pi}}\frac{1}{2t^{\frac{3}{2}}}\cos(t-\alpha) - \sqrt{\frac{2}{\pi}}\frac{1}{t^{\frac{1}{2}}}\sin(t-\alpha)$$

$$\frac{dI_n}{dt} = -\sqrt{\frac{2}{\pi}}\frac{1}{2t^{\frac{3}{2}}}\cos(t-\beta) - \sqrt{\frac{2}{\pi}}\frac{1}{t^{\frac{1}{2}}}\sin(t-\beta)\ ;$$

hence

$$U_t = \frac{2}{\pi}\sin(\alpha-\beta) = \frac{2}{\pi}\sin\frac{m-n}{2}\pi,$$

whilst for $t=0$ we arrive at $U_t = 0$.

So we find

$$\int_0^\infty \frac{I_m I_n}{t}\,dt = \frac{2}{\pi}\,\frac{\sin\frac{m-n}{2}\pi}{m^2-n^2}, \qquad m \neq n$$

and as a special case

$$\int^\infty \frac{I_n^2}{t}\,dt = \frac{1}{2n}.$$

From this formula many others may be deduced important for the theory of Bessel's functions.

Mathematics. — Mr. K. Bes: "*Analytical determination of the ninth point in which two curves of degree three, passing through eight given points, intersect each other.*" (Communicated by Prof. J. Cardinaal).

Let

$$\left.\begin{array}{l} a_1x^3+a_2x^2y+a_3x^2z+a_4xy^2+a_5xyz+a_6xz^2+a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3=0,\\ b_1x^3+b_2x^2y+b_3x^2z+b_4xy^2+b_5xyz+b_6xz^2+b_7y^3+b_8y^2z+b_9yz^2+b_{10}z^3=0, \end{array}\right\}(1)$$

7*