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Physics. — Mr. FRED. SCHUH on: "*Plane waves of light in an homogeneous, electrically and magnetically anisotropic dielectric.*" (2nd part).

10. Before examining the wave surface more closely, I shall first show that the ray of light is normal to the electric and the magnetic force, and therefore to the ray-plane. For this purpose we first show that the ray is electrically conjugate to $\mathfrak{D}$ and magnetically to $\mathfrak{B}$, properties which continue to exist after a transformation such as we have used. From equation (59) follows by (56), (57) and (58),

$$\frac{a_x\, l'\, \lambda'}{a_x\, v^2 - s'^2} + \frac{a_y\, m'\, \mu'}{a_y\, v^2 - s''^2} + \frac{a_z\, n'\, \nu'}{a_z\, v^2 - s''^2} = 0,$$

from which by (50), (51) and (52)

$$a_x f'\, \lambda' + a_y\, g'\, \mu' + a_z\, h'\, \nu' = 0,$$

which expresses that $\mathfrak{D}'$ and the transformed ray are conjugate diameters of the transformed electric ellipsoid.

If we take (50) into account, we derive from (56),

$$\lambda'\, a' = \frac{s''}{\varrho''}\, l'\, a' - \frac{s''\,(\varrho''^2 - s''^2)}{\varrho''\, v^2\,(a_x\, l'\, f' + a_y\, m'\, g' + a_z\, n'\, h')} f'\, a'.$$

Adding to this the corresponding equations for $\mu'\, b'$ and $\nu'\, c'$, we find:

$$\lambda'\, a' + \mu'\, b' + \nu'\, c' = 0,$$

which means that $\mathfrak{B}'$ and the transformed ray are conjugate diameters of the transformed magnetic ellipsoid (sphere with radius 1).

11. The electric force being normal to the magnetic induction and to the ray, we get:

$$\frac{P}{\nu\, b - \mu\, c} = \frac{Q}{\lambda\, c - \nu\, a} = \frac{R}{\mu\, a - \lambda\, b} = \frac{1}{A}.$$

If we substitute this in one of the equations (30), (31) and (32), we find:

$$A = \frac{v}{s}(l\lambda + m\mu + n\nu).$$

The expressions $l\lambda + m\mu + n\nu$ and $\frac{s}{\varrho}$ both being the cosine of the angle between the ray and the normal to the wave-front, so that:

$$l\lambda + m\mu + n\nu = \frac{s}{\varrho}, \quad \ldots \ldots \ldots \quad (61)$$

we find $A = \frac{v}{\varrho}$; consequently:

$$P = \frac{\varrho}{v}(\nu b - \mu c), \quad \ldots \ldots \ldots \quad (62)$$

$$Q = \frac{\varrho}{v}(\lambda c - \nu a), \quad \ldots \ldots \ldots \quad (63)$$

$$R = \frac{\varrho}{v}(\mu a - \lambda b), \quad \ldots \ldots \ldots \quad (64)$$

or

$$\mathfrak{E} = \frac{\varrho}{v} \underline{sin}_{\pi} (\underline{1}_{s} \mathfrak{B}). \quad \ldots \ldots \ldots \quad (65)$$

In the same way we deduce:

$$\alpha = \frac{\varrho}{v}(\mu h - \nu g), \quad \ldots \ldots \ldots \quad (66)$$

$$\beta = \frac{\varrho}{v}(\nu f - \lambda h), \quad \ldots \ldots \ldots \quad (67)$$

$$\gamma = \frac{\varrho}{v}(\lambda g - \mu f), \quad \ldots \ldots \ldots \quad (68)$$

or

$$-\mathfrak{H} = \frac{\varrho}{v} \underline{sin}_{\pi} (\underline{1}_{s} \mathfrak{D}). \quad \ldots \ldots \ldots \quad (69)$$

These equations are the same as (27) to (32), with the exception that $\mathfrak{D}$ and $\mathfrak{E}$ and in the same way $\mathfrak{B}$ and $\mathfrak{H}$ have been interchanged, and that $\frac{v}{s}$ is replaced by $\frac{\varrho}{v}$. By changing in the same way (39), (40), (41) and (42), we find:

$$\lambda = \frac{v}{\varrho} \, \frac{Q\gamma - R\beta}{4\pi U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$\mu = \frac{v}{\varrho} \, \frac{R\alpha - P\gamma}{4\pi U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$\nu = \frac{v}{\varrho} \, \frac{P\beta - Q\alpha}{4\pi U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$4\pi U \frac{\varrho}{v} \frac{1}{\ }_{s} = \frac{sin}{\ }_{\pi} (\mathfrak{H}\,\mathfrak{E}), \quad . \quad . \quad . \quad . \quad . \quad (73)$$

from which last equation we further deduce:

$$\varrho^2 = v^2 \frac{\mathfrak{H}^2\,\mathfrak{E}^2\, sin^2 (\mathfrak{H}\,\mathfrak{E})}{(4\pi U)^2} = v^2 \, sin^2 (\mathfrak{H}\,\mathfrak{E}) \, \frac{\mathfrak{E}}{\mathfrak{D} \cos (\mathfrak{D}\,\mathfrak{E})} \; \frac{\mathfrak{H}}{\mathfrak{B} \cos (\mathfrak{B}\,\mathfrak{H})},$$

and so, according to (9) and (18),

$$\varrho = v \, \underset{\mathfrak{E}}{r'_e} \; \underset{\mathfrak{H}}{r'_m} \; sin (\mathfrak{H}\,\mathfrak{E}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

This equation expresses that ϱ is equal to v times the area of the parallelogram, described on the radii vectores of the reciprocal electric and the reciprocal magnetic ellipsoid, resp. in the directions of $\mathfrak{E}$ and $\mathfrak{H}$.

The two values of ϱ for the same ray are equal only when the ray-plane is a section of similitude of the two reciprocal ellipsoids, and the electric and magnetic force are accordingly indeterminate in it.

12. By Poynting's theorem the flow of energy is greatest through a plane through $\mathfrak{H}$ and $\mathfrak{E}$, i.e. through the ray-plane. The amount of energy W, which according to that theorem flows through the ray-plane per unit of time and per unit of area is:

$$W = \frac{v}{4\pi} \mathfrak{H} \mathfrak{E} \sin(\mathfrak{H} \mathfrak{E}),$$

or, by (73)

$$W = U\varrho, \quad \ldots \ldots \ldots \quad (75)$$

a result which was to be expected.

13. Let us now examine the wave-surface somewhat closer. The section with a plane of coordinates degenerates into 2 ellipses, which are e.g. for $x = 0$:

$$\frac{y^2}{\zeta_e^2} + \frac{z^2}{\eta_e^2} = \frac{v^2}{\varepsilon_m^2 \zeta_e^2 \eta_e^2 w^2}$$

and

$$\frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = \frac{v^2}{\varepsilon_e^2 \zeta_m^2 \eta_m^2 w^2}.$$

The first ellipse is similar to the section of the electric ellipsoid $\frac{x^2}{\varepsilon_e^2} + \frac{y^2}{\zeta_e^2} + \frac{z^2}{\eta_e^2} = 1$ with the plane $x = 0$, and the second ellipse to the section of the magnetic ellipsoid $\frac{x^2}{\varepsilon_m^2} + \frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = 1$ with the plane $x = 0$. I shall call the first ellipse an electric ellipse and the second a magnetic one. The same applies to the sections with the planes $y = 0$ and $z = 0$. It is easy to find that the electric ellipse in one plane of coordinates intersects the magnetic ellipse situated in another plane of coordinates (of course in a point of a coordinate axis). If $b_1 > b_2 > b_3$, the electric ellipse in the YZ-plane lies quite outside, and that in the XY-plane quite inside the magnetic ellipse, while in the XZ-plane the two ellipses intersect in 4 points. These four points are conic points of the wave-surface. (It is easy to find analytically that the wave-surface can only have conic points in the three planes of coordinates and in the plane at infinity, which projectively may also be considered as plane of coordinates. The section of the wave-surface with each of these four planes degenerates into 2 conic sections; so that every plane furnishes four conic points; in all, 16 conic points, of which however only the four lying in the XZ plane are real. The wave-surface intersects the plane at infinity along the sections

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of the electric and the magnetic ellipsoid with that plane). The wave-surface having a finite number of conic points (16), it cannot degenerate into 2 other surfaces if $b_1 > b_2 > b_3$. The two sheets cohere in the 4 real conic points.

If $b_2 = b_3$, the wave-surface degenerates into two ellipsoids

$$\frac{x^2}{\varepsilon_e^2} + \frac{y^2}{\zeta_e^2} + \frac{z^2}{\eta_e^2} = \frac{v^2}{\varepsilon_e^2\,\zeta_e^2\,\eta_m^2\,w^2}$$

and

$$\frac{x^2}{\varepsilon_m^2} + \frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = \frac{v^2}{\varepsilon_m^2\,\zeta_e^2\,\eta_m^2\,w^2},$$

the first of which is similar to the electric, the second to the magnetic ellipsoid, and which I therefore call the electric and the magnetic part of the wave-surface. The two ellipsoids intersect the X-axis in the same point and touch each other in that point. The electric part of the wave-surface lies inside or outside the magnetic part, according to whether $b_1 > b_2$ or $b_1 < b_2$.

If $b_1 = b_2 = b_3$, the two ellipsoids coincide.

14. Let us return to the case $b_1 > b_2 > b_3$. We have seen that the ray of light is electrically conjugate to $\mathfrak{D}$ and magnetically to $\mathfrak{B}$. The ray of light is therefore the line of intersection of two planes; the first of these is electrically conjugate to $\mathfrak{D}$, and passes through $\mathfrak{B}$ and the line g_e which is electrically conjugate to the wave-front; the second is magnetically conjugate to $\mathfrak{B}$, and passes through $\mathfrak{D}$ and the line g_m, which is magnetically conjugate to the wave-front.

If a point of the wave-surface is given, so that the wave-front and the ray is known, we find $\mathfrak{D}$ and $\mathfrak{B}$ by letting planes pass through the ray and resp. through g_m and g_e, and making these planes intersect with the wave-front. We may also use the planes S_e and S_m which are electrically and magnetically conjugate to the ray and which intersect the wave-front resp. along $\mathfrak{D}$ and $\mathfrak{B}$. If the ray is electrically conjugate to the wave-front, the first construction fails for $\mathfrak{B}$ and the second for $\mathfrak{D}$, so that we can still apply one of the two constructions in order to find $\mathfrak{B}$ and $\mathfrak{D}$. (If the ray is magnetically conjugate to the wave-front, the reverse takes place). $\mathfrak{B}$ is then doubly conjugate to $\mathfrak{D}$ and to the ray and must therefore be a principal direction. The ray will consequently lie in a plane through two principal directions, and so also the point of the wave-surface; since ray and wave-front are electrically conjugate, this point will lie on the electric ellipse. $\mathfrak{B}$ falls now along a principal direction and $\mathfrak{D}$ touches the electric ellipse, as follows from the construction. Both constructions for $\mathfrak{D}$ and $\mathfrak{B}$ fail only when the ray is both electrically and magnetically conjugate to the wave-front, and falls therefore along a principal direction; the wave-front passes then through the two other principal directions. By now paying attention to the adjacent points of the wave surface, specially to points lying in the planes of coordinates, we find that $\mathfrak{D}$ touches the electric ellipse and $\mathfrak{B}$ the magnetic ellipse in the planes of coordinates.

15. To every wave-front belong two rays; if I and II are the directions of the possible electric and magnetic induction in the wave-front, the two rays of light are: the line of intersection of the planes through I and g_e and through II and g_m and the line of intersection of the planes through I and g_m and through II and g_e.
The question might be raised: When do the two rays of light fall in the same direction? Evidently when the ray of light is doubly conjugate both to I and to II and accordingly is a principal direction. We find also that this is the only case in which the two wave-fronts belonging to one direction of the ray coincide. The wave-front passes then through the two other principal directions.

16. Let us now examine the case of the wave-front being a section of similitude of the electric and the magnetic ellipsoid. The two lines g_e and g_m which resp. are electrically and magnetically conjugate to the wave-front, now lie both in the plane through two principal directions, viz. the X-axis and the Z-axis, the wave-front passing through the middle principal direction, the Y-axis; and they do not coincide. The ray being the line of intersection of the plane through $\mathfrak{B}$ and g_e and the plane through $\mathfrak{D}$ and g_m, while $\mathfrak{B}$ and $\mathfrak{D}$ are indeterminate, we get a cone of rays passing through g_e and g_m; for, if $\mathfrak{D}$ falls in the XZ-plane and $\mathfrak{B}$ along the Y-axis, g_e becomes the ray, and if $\mathfrak{B}$ falls in the XZ-plane and $\mathfrak{D}$ along the Y-axis, g_m becomes the ray. Moreover it is easily seen that these are the only rays falling in the XZ-plane and that therefore the cone is quadratic. If inversely the ray is given in this case, $\mathfrak{D}$ and $\mathfrak{B}$ may be found by means of one of the two given constructions. But whatever the course of the ray may be, we have always the same value of s, so that we have to deal with but one tangentplane to the wave-surface. This must touch along a curve (which is of course

an ellipse, as it counts as a double line of intersection and as the whole intersection with the wave-surface is of the 4th degree), which by its radii vectores through O indicates the possible rays of light. The tangent plane just mentioned touches the electric and the magnetic ellipses in the XZ-plane in the points A and B, and is parallel to the Y-axis; the rays g_e and g_m are the radii vectores of these points of contact, so that it is directly to be seen from the wave-surface that they belong to the cone of rays. Let now the ray be given by a point C of the ellipse of contact, then we find $\mathfrak{D}$ as the intersection of the wave-front with the plane through g_m and the ray, so that CB indicates the direction of $\mathfrak{D}$; in the same way CA indicates the direction of $\mathfrak{B}$. $\mathfrak{D}$ and $\mathfrak{B}$ being also conjugate diameters of the section of the wave-front with the electric and the magnetic ellipsoid, it follows directly, that the curve of contact must be similar to these elliptic sections with the same direction of axes. This might be seen, even if we did not yet know that the curve of contact is an ellipse. Further AB is a diameter of the ellipse of contact. (Internal conic refraction).

17. Let now the ray-plane be a section of similitude of the two reciprocal ellipsoids, then it passes through the middle reciprocal principal direction, so that the ray lies in the XZ-plane. Indeed, the ray of light is now the radius vector of one of the conic points of the wave-surface, and these points are only to be found in the XZ-plane. The wave-front is now indeterminate, being a tangent plane to the wave-surface in the conic point. (It is a quadratic conic point; else the line which connects it with a second conic point would have more than four points in common with the wave-surface). In a similar way as in the preceding case we can now show that the planes S_e and S_m which are electrically and magnetically conjugate to the ray, belong to the possible wave-fronts. This is also directly seen from the wave-surface, as the planes S_e en S_m are both parallel to the Y-axis, and, if transferred to the conic point, touch respectively the electric and the magnetic ellipse in the XZ-plane and so also the tangent cone in the conic point. By their intersection with the wave-front these planes S_e and S_m indicate directly the directions of $\mathfrak{D}$ and $\mathfrak{B}$. If the wave-front coincides with S_e, $\mathfrak{D}$ falls in the XZ-plane and touches the electric ellipse; $\mathfrak{B}$ is then parallel to the Y-axis. Similarly with what we have found, when the wave-front is determinate but the ray in-

determinate, we now get what follows. Let from O perpendiculars be drawn to all possible wave-fronts belonging to the radius vector of a conic point as ray. Let this cone of perpendiculars be intersected by a plane normal to the ray, which plane intersects the perpendiculars on S_e and S_m in A and B; let C be a point of that intersection, so that OC is the normal to the wave-front, then CB indicates the direction of the electric and CA the direction of the magnetic force. The intersection is therefore an ellipse, of which AB is a diameter and which is similar to the sections of the ray-plane with the two reciprocal ellipsoids, and has its axes in the same directions as these sections. (External conic refraction).

18. We see directly from the wave-surface that the ray for which the wave-front is indeterminate, and the wave-front for which the ray is indeterminate, do not belong together as ray and wave-front. The phenomena of internal and external conic refraction are therefore wholly separated from each other. The ellipse of contact encloses a conic point of the wave-surface. This ellipse is a spinodal curve of the wave-surface; the tangent-plane in one of its points intersects the surface in a curve with a double point and two coinciding tangents, in such a way however that the curve has not a cusp in the point of contact, as is generally the case on a spinodal curve, but that it consists of two coinciding curves. The surface is everywhere convex-convex, the concave side turned to O, except between the four conic points and the ellipses of contact, where the wave-surface is concave-convex.

19. The existence of tangent planes which touch along an ellipse, and which we may call ellipse-tangent planes, can also be directly derived from that of the conic points, if we remember that the surface is dualistic with itself. As we have four conic points in the XZ-plane, so we have four ellipse-tangent planes through the point whose point-coordinates are equal to the plane-coordinates of the XZ-plane, i. e. through the point at infinite distance on the Y-axis; these planes will be parallel to the Y-axis. There are also four ellipse-tangent planes parallel to the X- and four parallel to the Z-axis, which are, however, unreal. Four unreal ellipse tangent planes through O correspond to the four unreal conic points at infinite distance.

20. If $b_1 \lessgtr b_2 = b_3$, everything is much simpler. If the motion of light is given e.g. by a point P of the electric part of the wave-surface, which is not the point of intersection with the X-axis, we find that $\mathfrak{D}$ lies in the plane through P and the X-axis, and touches therefore the meridian through P; $\mathfrak{B}$ lies in a plane parallel to the YZ-plane and so touches the parallel-ellipse. The reverse takes place when P lies on the magnetic part of the wave-surface. If however, P lies on the X-axis, i. e. in the point where the two parts of the wave-surface touch each other, the ray is the X-axis and the wave-front the YZ-plane, $\mathfrak{D}$ and $\mathfrak{B}$ being now indeterminate in the wave-front (they must, however, be doubly conjugate to each other).

If finally $b_1 = b_2 = b_3$, the two parts of the wave-surface coincide. In this case to every wave-front belongs one ray and vice versa, $\mathfrak{D}$ and $\mathfrak{B}$ being always indeterminate in the wave-front.

Physics. — Mr. PH. KOHNSTAMM and Mr. B. M. VAN DALFSEN: "*Vapour-tensions of mixtures of ether and chloroform*". (Communicated by Prof. J. D. VAN DER WAALS).

For our determinations of vapour-tensions for mixtures of ether and chloroform we have made use of the dynamical method, i. e. we have determined the boiling-point at a certain pressure. As the methods of the determination of the vapour-tension, and specially the apparatus used by us, will be the subject of an extensive communication by one of us, which will appear before long, it seems superfluous to discuss these two points at length. Yet we will point out, specially to show how far our values are to be trusted, that we found it impossible to attain an accuracy greater than 1 m.m. mercury for dynamical determinations of vapour-tensions of mixtures. The values given are therefore at the utmost only in so far accurate; the errors of some of the observations can even become three or four times the amount. This is specially due to two sources of error, first the hydrostatic pressure of the boiling liquid, the influence of which was already pointed out by Dr. SMITS in the reports of these proceedings [1]), and secondly the superheating. We have tried to annul the disturbing

[1]) Volume II p. 475.