Citation:

Stein, J., On J.C.Kapteyn's criticism of AIRY's method to determine the Apex of the solar motion, in:
KNAW, Proceedings, 4, 1901-1902, Amsterdam, 1902, pp. 221-232

The relations (1) show a remarkable analogy to those existing between the values of the rotation α and α' about a and a' and the rotation round the central axis, the latter evidently amounting to ω.

We have namely:

$$\alpha = \omega \sin (ca'),$$

$$\alpha' = \omega \sin (ca).$$

Comparing these relations to (1), we see that $\angle OAA' = \angle (ca')$.

Now the plane (Aa') being the polar plane of A, it follows that the rotation of this plane, when A describes the line $O'A$, is equal to the rotation of OA: in other words, the rotation of a is equal to that of OA.

If now we imagine a line passing through O parallel to a, we immediately see that when A describes the line $O'A$, the plane (Oa) assumes a motion originating from a double rotation with equal components about the plane OAA' and the plane normal to it through O.

If finally we make the thus generated system of planes turn about the plane through O and the central axis c, we obtain the complete image of the reduction.

The results arrived at here entirely agree with those of Dr. W. A. WYTHOFF in his dissertation: „De Biquaternion als bewerking in de ruimte van vier afmetingen."

Astronomy. — "*On* J. C. KAPTEYN's *criticism of* AIRY's *method to determine the Apex of the solar motion,*" By J. STEIN S.J. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

At the meeting of the Section of Sciences of Jan. 27th 1900, Prof. J. C. KAPTEYN has given some critical remarks on the methods followed till now to determine the co-ordinates of the Apex of the solar motion. In his paper the writer would point out: first, that neither AIRY's nor ARGELANDER's method is based on the known hypothesis on the proper motions: "the peculiar proper motions of the fixed stars have no preference for any particular direction." Secondly he has tried to develop a method satisfying this condition. (Proceedings Vol. II, pag. 353).

It seems to me that this charge against AIRY's method is unjust; and I hold that this method, even when the equations of condition are treated with least squares, remains in perfect harmony with the hypothesis mentioned.

For a better understanding of the question it may perhaps be useful to give here in short AIRY's reasoning.

AIRY resolves the apparent proper motion into two axes at right angles to each other, and represents the components by the sum of the components of the parallactic motion of the sun, of the error of observation, of the error in the precessional constants and of the motus peculiaris.

Let T and U be the directions of those axes, M that of the motus peculiaris, H that of the Antapex, τ_0 and v_0 the components of the proper motion of a star, t and u the components of the errors of observation, m the linear motus peculiaris, h the linear motus parallacticus and ϱ the distance from the sun, then we have, omitting the correction of the precessional constants, the equations:

$$\left.\begin{aligned}\tau_0 &= \frac{h}{\varrho}\cos(H, T) + \frac{m}{\varrho}\cos(M, T) + t \\ v_0 &= \frac{h}{\varrho}\cos(H, U) + \frac{m}{\varrho}\cos(M, U) + u\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (A)$$

If we resolve the parallactic motion of the sun into three directions at right angles to each other, and we substitute

$$X\cos(X, U) + Y\cos(Y, U) + Z\cos(Z, U) \text{ for } h\cos(H, U)$$

$$X\cos(X, T) + Y\cos(Y, T) + Z\cos(Z, T) \text{ for } h\cos(H, T)$$

then each star will give two equations for the determination of X, Y and Z.

As however the relations between the error of observation and the motus peculiaris are not known, AIRY proposes two different solutions of these equations: 1°. on the supposition that the irregularities of proper motion are entirely due to errors of observation; 2°. that they are entirely due to peculiar motions of the stars. In either solution he supposes that the errors of observation or the motus peculiares respectively may be considered as chance-errors, and hence he solves the equations in both cases so, that either the sum of the squares of the errors of observation or the sum of the squares of the motus peculiares is a minimum.

We confine ourselves to the second supposition, and therefore give to each equation — ceteris paribus — the same weight. In the first supposition the weights ought to be proportional to ϱ^2. As we suppose with KAPTEYN (Proceedings p. 357) that the distances from the sun to the stars, whose motions are considered, as equal, the two sets of normal equations are identical on both the suppositions.

The three normal equations for X, Y and Z, proposed by AIRY, are derived from the equations for the two components τ and v. If however for some stars one of the components is unknown, we can deduce three normal equations from the other component, or if both components are known for all stars, we can, starting from each of the components separately, construct two sets each of three normal equations. KAPTEYN follows the last method, and so shall we.

It is of course immaterial what are the directions T and U of the components of the proper motion and we may choose those that are the most appropriate. AIRY uses the direction towards the north pole of the equator and the direction of the parallel; KAPTEYN chooses the direction towards a point near the Antapex and the direction at right angles to it; we shall also follow the latter method.

2. Meaning of the symbols according to KAPTEYN:

A_0 and D_0 right ascension and declination of the assumed Antapex;
λ_0 the distance from the star to this point;
χ_0 the angle made by the declination circle with the direction towards this point;
v_0 the component of the total proper motion μ according to the latter direction;
τ_0 the component perpendicular to the preceding;
p_0 the angle, made by the total proper motion with the parallactic proper motion.

The symbols without index $(_0)$ will be used when the real, instead of the assumed, Antapex is meant.

Let ε be the angle made by the directions of the star towards the assumed and the real Antapex, then we can put the equations (A) in this form:

$$\tau_0 = \mu \sin p_0 = \frac{h}{\varrho} \sin \lambda \sin \varepsilon ; \quad v_0 = \mu \cos p_0 = \frac{h}{\varrho} \sin \lambda \cos \varepsilon \quad . \quad . \quad (A')$$

The two last terms of the equations for τ_0 and v_0 are both considered as chance-errors of observation, and hence are left out of consideration.

Now

$$\varepsilon = - d\chi_0 = - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD,$$

$$\lambda = \lambda_0 + \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD,$$

and neglecting small quantities of higher order we can put:

$$\tau_0 = \frac{h}{\varrho} \sin \lambda_0 \left\{ - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\};$$

$$v_0 = \frac{h}{\varrho} \sin \lambda_0 + \frac{h}{\varrho} \cos \lambda_0 \left\{ \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD \right\}.$$

Here we assume as unknown quantities $\frac{h}{\varrho}$, $\frac{h}{\varrho} dA$ and $\frac{h}{\varrho} dD$, and obtain from the equations for τ_0 the two following normal equations:

$$\left.\begin{aligned}
\left[-\tau_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \sin \lambda_0\right] &= \left[\left(\frac{\partial \chi}{\partial A}\right)_0^2 \sin^2 \lambda_0\right] \frac{h}{\varrho} dA + \left[\left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin^2 \lambda_0\right] \frac{h}{\varrho} dD \\
\left[-\tau_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin \lambda_0\right] &= \left[\left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin^2 \lambda_0\right] \frac{h}{\varrho} dA + \left[\left(\frac{\partial \chi}{\partial D}\right)_0^2 \sin^2 \lambda_0\right] \frac{h}{\varrho} dD
\end{aligned}\right\} (B)$$

From the equation for v_0 we derive three normal equations, of which the first is:

$$\left[v_0 \sin \lambda_0\right] = \left[\sin^2 \lambda_0\right] \frac{h}{\varrho} + \left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0\right] \frac{h}{\varrho} dA +$$
$$+ \left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0\right] \frac{h}{\varrho} dD$$

The two other equations are left out of consideration on account of their small weight.

For stars, distributed symmetrically with regard to the Apex and the Antapex, both $\left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0\right]$ and $\left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0\right]$

are equal to zero. The same holds for stars in the same great circle passing through the assumed Antapex at distances of λ_0 and $180-\lambda_0$ from that point. Hence, when the stars are equally scattered over the heavens

$$\left[\sin\lambda_0\cos\lambda_0\left(\frac{\partial\lambda}{\partial A}\right)_0\right] \text{ and } \left[\sin\lambda_0\;\cos\lambda_0\left(\frac{\partial\lambda}{\partial D}\right)_0\right]=0.$$

When the stars are unequally distributed, these two values will yet be small with regard to $\left[\sin^2\lambda_0\right]$. Moreover $\frac{h}{\varrho}dA$ and $\frac{h}{\varrho}dD$ are small quantities with regard to $\frac{h}{\varrho}$, when the error in the assumed Apex is small. If however after a first calculation it would appear that dA and dD were not so small that we could neglect the quantities of the second order, the calculation must be repeated with more accurate values of A_0 and D_0; in this case the two last terms of the normal equations may be neglected with regard to $\left[\sin^2\lambda_0\right]\frac{h}{\varrho}$.

We then obtain:

$$[v_0\sin\lambda_0]=[\sin^2\lambda_0]\frac{h}{\varrho} \text{ or } \frac{h}{\varrho}=\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}.$$

If this value of $\frac{h}{\varrho}$ *differs from zero*, it may be substituted in the equations (B), and then the determination of dA and dD depends on the solution of:

$$\left.\begin{aligned}\left[-\tau_0\left(\frac{\partial\chi}{\partial A}\right)_0\sin\lambda_0\right]&=\left[\left(\frac{\partial\chi}{\partial A}\right)_0^2\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dA+\\&\quad+\left[\left(\frac{\partial\chi}{\partial A}\right)_0\left(\frac{\partial\chi}{dD}\right)_0\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dD\\\left[-\tau_0\left(\frac{\partial\chi}{\partial D}\right)_0\sin\lambda_0\right]&=\left[\left(\frac{\partial\chi}{\partial A}\right)_0\left(\frac{\partial\chi}{\partial D}\right)_0\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dA+\\&\quad+\left[\left(\frac{\partial\chi}{\partial D}\right)_0^2\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dD\end{aligned}\right\}(B')$$

If we have started from the correct Apex, dA and dD are both equal to zero; therefore if the Apex is determined according to AIRY's method, the conditions:

$$\left[\tau \frac{\partial \chi}{\partial A} \sin \lambda\right] = 0 \text{ and } \left[\tau \frac{\partial \chi}{\partial D} \sin \lambda\right] = 0 \quad . \quad . \quad . \quad . \quad (C)$$

must be satisfied. The same conditions have been deduced by KAPTEYN from his fundamental hypothesis (p. 359).

If however $\frac{h}{\varrho} = 0$, the coefficients of dA and dD in the equations (B) are zero, and further Apex-determination is out of the question. As a first objection against KAPTEYN's normal equations may be mentioned that it is not self-evident that a solution of his equations is impossible in this case; on the contrary, with a given combination of τ and v, the position of the non-existing Apex may be arrived at.

3. We shall now try to prove that the conditions which, according to KAPTEYN (Proceedings p. 362), may be derived from AIRY's method are not correct.

When the position of the Apex and the amount of the solar motion have been found, and the apparent proper motion is resolved into the peculiar proper motion and the parallactic one, the sum of the squares of the components of the peculiar proper motion, according to AIRY, must be a minimum. As the component of the parallactic solar motion perpendicular to the direction of the true Apex is zero, the place of the Apex and the amount of the solar motion must be determined so, that:

$$[\tau^2] = \text{minimum and } \left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right] = \text{minimum}$$

(see KAPTEYN l. c.)

Let q be the angle made by the motus peculiaris m with the direction of the star towards the true Antapex, whose Right Asc. and Decl. are A and D then:

$$\tau = \frac{m}{\varrho} \sin q.$$

If, however, we resolve the proper motion into two components, one in a direction towards a point outside the Apex (Right Asc.

$A + dA$ Decl. $D + dD$), and one in a diıection at right angles to it, the latter will be

$$\tau_0 = \frac{m}{\varrho} \sin (q+\varepsilon) + \frac{h}{\varrho} \sin (\lambda+d\lambda) \sin \varepsilon,$$

when ε represents the small angle made by the direction towards the true Apex with the direction towards $A + dA$, $D + dD$, or neglecting small quantities:

$$\tau_0 = \tau + \frac{m}{\varrho} \cos q . \varepsilon + \frac{h}{\varrho} \sin \lambda . \varepsilon .$$

In order that $[\tau^2]$ may be a minimum

$$\left[\tau \frac{m}{\varrho} \cos q . \varepsilon + \tau \frac{h}{\varrho} \sin \lambda . \varepsilon\right] \text{ must be zero.}$$

If we substitute for ε its value $- \frac{\partial \chi}{\partial A} dA - \frac{\partial \chi}{\partial D} dD$, we obtain ($dA$ and dD being independent quantities):

$$\left[\tau \frac{m}{\varrho} \cos q \frac{\partial \chi}{\partial A} + \tau \frac{h}{\varrho} \sin \lambda \frac{\partial \chi}{\partial A}\right] = 0 \text{ and } \left[\tau \frac{m}{\varrho} \cos q \frac{\partial \chi}{\partial D} + \tau \frac{h}{\varrho} \sin \lambda \frac{\partial \chi}{\partial D}\right] = 0$$

As, however, the motus peculiaris may be considered as an error of observation, which does not enter into the equations (A'), the equations of condition are reduced to:

$$\frac{h}{\varrho}\left[\tau \sin \lambda \frac{\partial \chi}{\partial A}\right] = 0 \text{ and } \frac{h}{\varrho}\left[\tau \sin \lambda \frac{\partial \chi}{\partial D}\right] = 0, \quad . \quad . \quad . \quad (C')$$

which shows that also the equation $[\varepsilon^2]$ = minimum leads to the right result; for, if $\frac{h}{\varrho}$ differs from zero, the conditions (C) and (C') are identical; if $\frac{h}{\varrho} = 0$, (C') assumes the indefinite form

$$\left[\tau \sin \lambda \frac{\partial \chi}{\partial A}\right] = \frac{0}{0}, \left[\tau \sin \lambda \frac{\partial \chi}{\partial D}\right] = \frac{0}{0}.$$

4. The reasoning which leads KAPTEYN to reject the condition $[\tau^2]=$ minimum, is as follows:

"$[\tau^2]$ is a minimum for

$$\left[\tau \frac{\partial \tau}{\partial A}\right]=0 \text{ and } \left[\tau \frac{\partial \tau}{\partial D}\right]=0,$$

and if we put

$$\frac{\partial \tau}{\partial A}=v\frac{\partial \chi}{\partial A},\ \frac{\partial \tau}{\partial D}=v\frac{\partial \chi}{\partial D},$$

the minimum conditions are:

$$\left[\tau\, v \frac{\partial \chi}{\partial A}\right]=0 \text{ and } \left[\tau\, v \frac{\partial \chi}{\partial D}\right]=0. \quad . \quad . \quad . \quad . \quad (D)$$

which differ from the right ones (C)."

It will be seen immediately, that the set (D) corresponds to the solution of the equations (one for each star):

$$\tau_0 = - v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA - v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD \quad . \quad . \quad . \quad . \quad (E)$$

It is therefore perfectly consequent to his reasoning, when KAPTEYN puts AIRY's relation in this form (Proceedings p. 369), differing from the form given by me

$$\tau_0 = \frac{h}{\varrho} \sin\lambda \left\{ -\left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\}.$$

KAPTEYN's equation (E) would be the right one, if AIRY had formulated his question thus: *To find a point so, that if it is connected with all the stars by means of great circles, the sum of the squares of the components of the proper motion, perpendicular to those circles, is a minimum — without considering the question whether a parallactic solar motion exists or not.* But this not being the principle of AIRY's method KAPTEYN's criticism of that method is incorrect.

5. The difference between the two considerations may also be put thus:

let A_0 and D_0 be the co-ordinates of a given point at the heavens; (if there is a parallactic solar motion, that point may be the assumed Antapex).

1°. If there existed only a parallactic solar motion, the proper motion for each separate star would be represented exactly by the formulae:

$$\tau_0 = \frac{h}{\varrho} \sin\lambda \sin\varepsilon ; \quad v_0 = \frac{h}{\varrho} \sin\lambda \cos\varepsilon ;$$

hence:

$$\frac{\partial \tau_0}{\partial A_0} = \frac{h}{\varrho} \sin\lambda \cos\varepsilon \frac{\partial \chi_0}{\partial A_0}, \quad \frac{\partial \tau_0}{\partial D_0} = \frac{h}{\varrho} \sin\lambda \cos\varepsilon \frac{\partial \chi_0}{\partial D_0} \text{ etc.}$$

2°. Even if the proper motions are distributed arbitrarily, without being influenced by any parallactic motion

$$\tau_0 = \mu \sin p_0, \quad v_0 = \mu \cos p_0,$$

$$\frac{\partial \tau_0}{\partial A_0} = v_0 \frac{\partial \chi_0}{\partial A_0}, \quad \frac{\partial \tau_0}{\partial D_0} = v_0 \frac{\partial \chi_0}{\partial D_0} \text{ etc.}$$

hold for each star separately.

AIRY starts from the first set, KAPTEYN from the second.

6. If we substitute in the conditions (C)

$$\tau = \tau_0 + \frac{h}{\varrho} \sin\lambda \left\{ - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\}$$

$$\lambda = \lambda_0 + \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD \text{ and } \frac{\partial \chi}{\partial A} = \left(\frac{\partial \chi}{\partial A}\right)_0 +$$

$$+ \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 dA + \left(\frac{\partial^2 \chi}{\partial A\, \partial D}\right)_0 dD \text{ etc.}$$

and neglect the small quantities, we obtain, the equations found before (B).

The equations which KAPTEYN (l.c.p.360) deduces from the same two conditions (C) differ from ours, because also in this case he uses the development

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$$\tau = \tau_0 + v_0\left(\frac{\partial \chi}{\partial A}\right)_0 dA + v_0\left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

which equation, in contradiction to that used by AIRY, is independent from the existence of a parallactic solar motion; therefore I hold myself authorized to consider AIRY's transformed equations (B) as corresponding more closely to the fundamental hypothesis than those of KAPTEYN.

7. The condition $\left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right] =$ minimum, may again serve to eliminate $\frac{h}{\varrho}$ from the equations (B).

As the position of the Apex and the amount of the solar motion are mutually independent, we consider:

1°. the relation which exists between $\left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right]$ and the position of the Apex.

If we augment the right ascension and the declination with dA and dD, we get v_0 for v, and $\lambda_0 = \lambda + d\lambda$ for λ.

Now

$$v = \frac{m}{\varrho} \cos q + \frac{h}{\varrho} \sin\lambda \text{ and } v_0 = \frac{m}{\varrho} \cos(q + \varepsilon) + \frac{h}{\varrho} \sin\lambda \cos\varepsilon$$

or:

$$v_0 = \frac{m}{\varrho} \cos q - \frac{m}{\varrho} \sin q \,.\, \varepsilon + \frac{h}{\varrho} \sin\lambda,$$

$$\text{while } \frac{h}{\varrho} \sin\lambda_0 = \frac{h}{\varrho} \sin\lambda + \frac{h}{\varrho} \cos\lambda \ d\lambda.$$

Hence:

$$\left[\left(v_0 - \frac{h}{\varrho} \sin\lambda_0\right)^2\right] = \left[\left(\frac{m}{\varrho} \cos q - \frac{m}{\varrho} \sin q \,.\, \varepsilon - \frac{h}{\varrho} \cos\lambda \ d\lambda\right)^2\right].$$

In order that $\left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right]$ or $\left[\left(\frac{m}{\varrho} \cos q\right)^2\right]$ may really be a minimum

$$\left[\frac{m}{\varrho} \cos q \,.\, \frac{m}{\varrho} \sin q \,.\, \varepsilon - \frac{m}{\varrho} \cos q \,.\, \frac{h}{\varrho} \cos\lambda \ d\lambda\right] \text{must be } 0.$$

But as in the case in hand the peculiar proper motion m is left out of consideration, this condition does not teach us anything about the position of the Apex.

2°. the relation which exists between $\left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right]$ and the amount of the solar motion $\frac{h}{\varrho}$.

$\left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right]$ must be always smaller than $\left[\left(v - \frac{h+dh}{\varrho} \sin \lambda\right)^2\right]$.

In order that it may be so,

$$\left[- v \frac{dh}{\varrho} \sin \lambda + \frac{h\, dh}{\varrho^2} \sin^2 \lambda\right] \text{ or } \left[- v \sin \lambda + \frac{h}{\varrho} \sin^2 \lambda\right] \text{ must be } 0,$$

whence follows

$$\frac{h}{\varrho} = \frac{[v \sin \lambda]}{[\sin^2 \lambda]}.$$

Thus, after the substitution of this value, we again arrive at the same normal equations (B') for the determination of dA and dD.

8. To conclude I remark that the equations derived in this paper become identical with those of KAPTEYN as soon as we confine ourselves to stars in one direction only. But even when we apply our theory to a great number of stars scattered over the heavens, the two sets will yield little differing results. For if we resolve v_0 into two parts $v_1 + v_2$, where $v_1 = \frac{h}{\varrho} \sin \lambda =$ the component of the parallactic solar motion, and $v_2 =$ the component of the peculiar proper motion, the coefficient of dA in the first of KAPTEYN's equations becomes

$$\left[v_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2\right] = \left[\frac{h}{\varrho} \sin^2 \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2\right] + \left[v_2 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2\right].$$

As according to the hypothesis there is an equal number of positive and negative values of v_2, the second term may be neglected, by which the coefficient becomes identical to the corresponding one in our set of equations (B). The same holds for the other coefficients.

It is superfluous to refute at large the objections against AIRY's method derived by KAPTEYN from a few particular cases of proper motion, because it seems to me that conclusions deduced from the consideration of only a few proper motions, chosen quite systematically, can hardly serve as criteria of a method which, as a matter of course, presupposes as data a great number of proper motions chosen at random. Finally attention must be drawn to an important point. In this paper (comp. § 1), following the method of KAPTEYN and others, I have considered separately the equations for τ and v.

Also in this modified form, as I have proved, AIRY's method leads to the right result. In AIRY's original method however, the three normal equations are composed from the equations for the two components τ and v. In this case there is but one equation of condition, viz.:

$$[m^2] \text{ or } [\tau^2] + \left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right] = \text{minimum},$$

i. e. "the direction and the amount of the parallactic motion must be chosen so, that *the sum of the squares of the* TOTAL *motus peculiares becomes a minimum.*" If this condition is applied to the instances given by KAPTEYN, it immediately becomes evident, that we arrive at the same Apex as KAPTEYN determines by applying the condition $[\tau = 0]$.

Astronomy. — *Reply to the criticism of* Dr. J. STEIN S.J. *by* J. C. KAPTEIJN.

It appears to be very probable that Dr. STEIN has not completely understood my paper in the proceedings of the February meeting of last year. This fact, and the fear that on the other hand I may also have misunderstood STEIN's reasoning (for one part at least of his paper this is certain) have led me to make my reply more circumstantial and elementary than might otherwise seem necessary.

With a view to the importance of the application of the method of least squares for the whole problem, it seems desirable to recall to mind the following elementary points relating to that method.

a). Let a system of equations of condition be given, thus:

$$\left.\begin{aligned} a_1 x + b_1 y &= n_1 \\ a_2 x + b_2 y &= n_2 \\ a_3 x + b_3 y &= n_3 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

$$\ldots\ldots\ldots$$