Mathematics. — "*A formula for the volume of the prismoid.*"
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As is known the volume of the prismoid is given by the expression

$$I = \frac{1}{6} h (P + Q + 4M),$$

where h denotes the distance of the parallel faces, P and Q represent the areas of these faces and M stands for the area of the section with the plane bisecting the distance of the parallel faces.

We wish to show here, that this formula forms a special case of a more general one, in which M has been replaced by the area D_q^p of the section with the plane dividing the distance h between P and Q in the ratio of p to q.

By joining any point O of the face P with all the vertices of the prismoid and combining these lines two by two by the necessary planes the prismoid is divided into: 1°. a pyramid with vertex O and base Q, 2°. a number of tetrahedrons with three vertices in P and one vertex in Q, 3°. a number of tetrahedrons with two vertices in each of the faces P, Q.

Let us first consider a tetrahedron $P_1 P_2 Q_3 Q_4$ of the last group and suppose that the edges $P_1 Q_3$, $P_1 Q_4$, $P_2 Q_3$, $P_2 Q_4$ meet any plane parallel to $P_1 P_2$ and $Q_3 Q_4$ in the points D_{13}, D_{14}, D_{23}, D_{24}. Then we have

$$D_{13} D_{23} = D_{14} D_{24} = \frac{q}{p+q} P_1 P_2 \text{ and } D_{13} D_{14} = D_{23} D_{24} = \frac{p}{p+q} Q_3 Q_4.$$

Now this tetrahedron can be considered as the second of three tetrahedrons, into which a certain prism with three side-faces is divided by two diagonal planes. Of any of the two triangles in the parallel planes limiting this prism two of the three sides are parallel to sides of the parallelogram $D_{13} D_{23} D_{24} D_{14}$; so the areas of these figures are in the ratio of $P_1 P_2 \times Q_3 Q_4$ to 2 times $D_{13} D_{23} \times D_{13} D_{14}$, i. e. as $(p+q)^2$ to $2pq$. So the volume of the tetrahedron is equal to

$$\frac{1}{3} h . \frac{(p+q)^2}{2pq} D',$$

D' representing the area of the parallelogram.

The pyramid with vertex O and base Q determines in the plane

D_q^p an area D'' equal to $\dfrac{p^2}{(p+q)^2}\,Q$. So the volume of this pyramid is equal to

$$\frac{1}{3}\,h.\frac{(p+q)^2}{2pq}\,D' + \frac{1}{3}\,h.\frac{2q-p}{2q}\,Q.$$

In the same manner we find that the volume of a tetrahedron of the second group can be represented by

$$\frac{1}{3}\,h.\frac{(p+q)^2}{2pq}\,D''' + \frac{1}{3}\,h.\frac{2p-q}{2p}\,P.$$

Moreover we have $\Sigma D' + D'' + \Sigma D''' = D_q^p$; so the volume of the entire prismoid is given by the formula

$$I = \frac{1}{6}\,h\left[\frac{2p-q}{p}\,P + \frac{2q-p}{q}\,Q + \frac{(p+q)^2}{pq}\,D_q^p\right] \quad . \quad . \quad (1)$$

For $p = q = 1$ we reobtain the result $I = \frac{1}{6}\,h\,(P + Q + 4\,M)$, as it ought to be. From the two formulae we deduce

$$(p+q)^2\,D_q^p = q\,(q-p)\,P + p\,(p-q)\,Q + 4\,pq\,M \quad . \quad . \quad (2)$$

For $p = 1$, $q = 2$ we find the remarkably simple relation

$$I = \frac{1}{4}\,h\,(Q + 3\,D_2^1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Still in another manner the volume can be expressed by means of *two* parallel sections. By interchanging p and q in (1) we get

$$I = \frac{h}{6\,pq}\left[(2\,pq - p^2)\,P + (2\,pq - q^2)\,Q + (p+q)^2\,D_p^q\right].$$

By addition of this equation to (1) we find

$$I = \frac{h}{12\,pq}\left[(4\,pq - p^2 - q^2)\,(P+Q) + (p+q)^2\,(D_q^p + D_p^q)\right] . \quad . \quad (4)$$

For $p = \sqrt{3} + 1$ and $q = \sqrt{3} - 1$ this relation gives finally

$$I = \frac{1}{2}\,h\left(D_{\sqrt{3}-1}^{\sqrt{3}+1} + D_{\sqrt{3}+1}^{\sqrt{3}-1}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$