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Physics. — H. A. LORENTZ. "*The rotation of the plane of polarization in moving media.*"

(Communicated in the meeting of March 29, 1902).

§ 1. In my „Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern” (Leiden, 1895) I examined the propagation of light in transparent bodies having a constant translation with velocity p , the aether being supposed to remain at rest, and tried to find, in how far optical phenomena may be affected by this motion. In the case of the rotation of the plane of polarization in optically active substances, I had to leave the question undecided. Indeed, the relation between the electric force \mathfrak{E} and the electric moment \mathfrak{M} , to which I was led by certain general principles (linear form of the equations, isotropy of structure and reversibility of the motions) does not only contain the coefficient j , which determines the rotation in the quiescent medium; there is besides a second coefficient k , which is multiplied by the velocity p , and whose ratio to j I could not determine, because I wished to refrain from special hypotheses as to the mechanism of the phenomenon.

The equation in question is¹⁾

$$\mathfrak{E} = \sigma \mathfrak{M} + j \text{Rot } \mathfrak{M} + k [\mathfrak{M} \cdot p], \dots \dots (1)$$

and the rotation for unit length was found to be²⁾

$$\omega = \frac{2\pi}{\sigma^2} n'^2 j,$$

if the body is at rest, and

$$\omega = \frac{2\pi}{\sigma^2} n'^2 \left(1 + \frac{W p_x}{c^2}\right) j + \frac{2\pi}{\sigma^2} n'^2 W p_x k, \dots \dots (2)$$

if it has a translation along the axis of x , the light travelling in the same direction. In these formulae n' is the frequency, i. e. the number of vibrations in the time 2π , for an observer, moving with the medium, W the mean of the velocities of right-handed and left-handed circularly polarized rays in the medium at rest, and c the velocity of light in the aether.

The two terms with p_x would annul each other, if

$$k = -\frac{j}{c^2} \dots \dots \dots (3)$$

I saw however no reason to admit this relation.

§ 2. Mr. LARMOR has published³⁾ some objections to my in-

¹⁾ l. c., p. 80. σ is the coefficient which determines the index of refraction.

²⁾ l. c., p. 118.

³⁾ J. LARMOR. *Aether and Matter*, Cambridge, 1900.

vestigation. According to him we may infer from theory that a translation has *no* influence on the rotation. Mr. LARMOR believes the contradiction between our results to be due to an error on my side¹⁾, consisting in an oversight which he points out p.p. 214 and 215 of his work.

A new examination of the problem has convinced me that LARMOR must be wrong in this assertion, the formula (2) following undoubtedly from my fundamental equations. I also found that the equations of LARMOR are the same as mine, if in these one puts $k=0$, and that it is only in consequence of a mistake that his analysis does not lead him to an expression agreeing with the first term in (2). I shall now show that, whereas my equations leave room for a compensation, just because they contain the second coefficient k , LARMOR, treating only the particular case $k=0$, ought to have arrived at a rotation, different for a moving and for a quiescent body.

§ 3. In my calculations I used the equations

$$\left. \begin{aligned}
Div \mathfrak{D} &= 0, \\
Div \mathfrak{H} &= 0, \\
Rot \mathfrak{H}' &= 4 \pi \mathfrak{D}, \\
Rot \mathfrak{E} &= - \dot{\mathfrak{H}}, \\
\mathfrak{E} &= 4 \pi c^2 \mathfrak{d} + [p, \mathfrak{H}], \\
\mathfrak{H}' &= \mathfrak{H} - 4 \pi [p, \mathfrak{d}], \\
\mathfrak{D} &= \mathfrak{d} + \mathfrak{M},
\end{aligned} \right\} \dots \dots \dots (4)$$

to which is to be added the relation (1).

The meaning of \mathfrak{E} and \mathfrak{M} has already been mentioned; \mathfrak{H} is the magnetic force and the remaining vectors are defined by the equations themselves. The components of the vectors are regarded as functions of the time and of the coordinates x, y, z , referred to axes, fixed to the moving medium; the time-rates of variation for constant values of these coordinates are denoted by $\dot{\mathfrak{D}}$ and $\dot{\mathfrak{H}}$.

We may omit the first and second formulae, these being implied, in the cases to be considered, in the third and fourth equations. Moreover, just like LARMOR, we shall restrict the investigation to bodies, moving parallel to the axis of x , and traversed by rays of light of this same direction. Then, the only independent variables are x and t , and the equations (4) become

¹⁾ l.c., p. 62.

$$\left. \begin{aligned}
 -\frac{\partial \mathfrak{H}'_z}{\partial x} &= 4\pi \dot{\mathfrak{D}}_y, & \frac{\partial \mathfrak{H}'_y}{\partial x} &= 4\pi \dot{\mathfrak{D}}_z, \\
 -\frac{\partial \mathfrak{E}_z}{\partial x} &= -\dot{\mathfrak{H}}_y, & \frac{\partial \mathfrak{E}_y}{\partial x} &= -\dot{\mathfrak{H}}_z, \\
 \mathfrak{E}_y &= 4\pi c^2 \mathfrak{d}_y - \nu_x \mathfrak{H}_z, & \mathfrak{E}_z &= 4\pi c^2 \mathfrak{d}_z + \nu_x \mathfrak{H}_y, \\
 \mathfrak{H}'_y &= \mathfrak{H}_y + 4\pi \nu_x \mathfrak{d}_z, & \mathfrak{H}'_z &= \mathfrak{H}_z - 4\pi \nu_x \mathfrak{d}_y, \\
 \mathfrak{D}_y &= \mathfrak{d}_y + \mathfrak{M}_y, & \mathfrak{D}_z &= \mathfrak{d}_z + \mathfrak{M}_z.
 \end{aligned} \right\} \dots (5)$$

§ 4. In LARMOR'S equations¹⁾ the velocity of translation ν_x is represented by v , the sign $\frac{\delta}{dt}$ is used for those time-rates of variation, which I have indicated by a dot, and the sign $\frac{d}{dt}$ for the differential coefficients relating to a fixed point of space. Hence, in his notation,

$$\frac{\delta}{dt} = \frac{d}{dt} + v \frac{d}{dx}.$$

If now, we write $\dot{\varphi}$ instead of $\frac{\delta \varphi}{dt}$, and $\varphi - v \frac{\partial \varphi}{\partial x}$ instead of $\frac{d\varphi}{dt}$ (φ being any quantity, depending on place and time), and if besides we suppose the substance to be unmagnetizable, so that $\mu = 1$, the equations of LARMOR become

$$\left. \begin{aligned}
 -\frac{\partial \gamma}{\partial x} &= 4\pi (\dot{g} + \dot{g}') - 4\pi v \frac{\partial (g + g')}{\partial x}, \\
 \frac{\partial \beta}{\partial x} &= 4\pi (\dot{h} + \dot{h}') - 4\pi v \frac{\partial (h + h')}{\partial x}, \\
 -\frac{\partial R}{\partial x} &= -\dot{b}, & \frac{\partial Q}{\partial x} &= -\dot{c},
 \end{aligned} \right\}$$

$$\begin{aligned}
 Q &= 4\pi c^2 g - v c, & R &= 4\pi c^2 h + v b, \\
 \beta &= b - 4\pi v h', & \gamma &= c + 4\pi v g'.
 \end{aligned}$$

These are the same as (5), as will be seen, if we replace

$$\begin{array}{l}
 Q, \quad R, \quad g, \quad h, \quad g', \quad h' \\
 \text{by} \\
 \mathfrak{E}_y, \quad \mathfrak{E}_z, \quad \mathfrak{d}_y, \quad \mathfrak{d}_z, \quad \mathfrak{M}_y, \quad \mathfrak{M}_z \\
 \text{and} \\
 b, \quad c, \quad \beta + 4\pi v (h + h'), \quad \gamma - 4\pi v (g + g') \\
 \text{by} \\
 \mathfrak{H}_y, \quad \mathfrak{H}_z, \quad \mathfrak{H}'_y, \quad \mathfrak{H}'_z.
 \end{array}$$

¹⁾ l. c., p. 212.

§ 5. As to the relation between electric polarization and electric force, this is given by LARMOR in the form ¹⁾

$$g' = \frac{K-1}{4\pi c^2} Q + \frac{\epsilon_2}{4\pi c^2} \frac{\partial R}{\partial x},$$

$$h' = \frac{K-1}{4\pi c^2} R - \frac{\epsilon_2}{4\pi c^2} \frac{\partial Q}{\partial x},$$

or

$$\left. \begin{aligned} \mathfrak{M}_y &= \frac{K-1}{4\pi c^2} \mathfrak{E}_y + \frac{\epsilon_2}{4\pi c^2} \frac{\partial \mathfrak{E}_z}{\partial x}, \\ \mathfrak{M}_z &= \frac{K-1}{4\pi c^2} \mathfrak{E}_z - \frac{\epsilon_2}{4\pi c^2} \frac{\partial \mathfrak{E}_y}{\partial x}. \end{aligned} \right\} \dots \dots (6)$$

Now, in my formula (1) the rotational terms are very much smaller than the first term $\sigma \mathfrak{M}$. We may therefore, in those terms, replace \mathfrak{M} by $\frac{1}{\sigma} \mathfrak{E}$. Hence

$$\mathfrak{M} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma^2} \text{Rot } \mathfrak{E} - \frac{k}{\sigma^2} [\dot{\mathfrak{E}} \cdot \mathfrak{p}], \dots \dots (7)$$

and, in the case under consideration,

$$\left. \begin{aligned} \mathfrak{M}_y &= \frac{1}{\sigma} \mathfrak{E}_y + \frac{j}{\sigma^2} \frac{\partial \mathfrak{E}_z}{\partial x} - \frac{k}{\sigma^2} \mathfrak{p}_x \dot{\mathfrak{E}}_z, \\ \mathfrak{M}_z &= \frac{1}{\sigma} \mathfrak{E}_z - \frac{j}{\sigma^2} \frac{\partial \mathfrak{E}_y}{\partial x} + \frac{k}{\sigma^2} \mathfrak{p}_x \dot{\mathfrak{E}}_y. \end{aligned} \right\} \dots \dots (8)$$

If this is compared with (6), it appears that the formulae of LARMOR agree with the particular case $k = 0$ of my theory, and that the coefficients we have introduced are related to each other as follows:

$$\frac{K-1}{4\pi c^2} = \frac{1}{\sigma}, \quad \frac{\epsilon_2}{4\pi c^2} = \frac{j}{\sigma^2} \dots \dots (9)$$

§ 6. For $k = 0$ my formula (2) gives

$$\omega = \frac{2\pi}{\sigma^2} n'^2 \left(1 + \frac{W \mathfrak{p}_x}{c^2} \right) j, \dots \dots (10)$$

a value depending on \mathfrak{p}_x . On the contrary, LARMOR's result does not contain the velocity of translation, but this is only so, because his calculation of the angle of rotation is not quite exact.

As is well known, this angle may be expressed in the velocities of propagation of right- and left-handed circularly polarized rays. In doing this, we have first of all to assign to the period of vibration, taken with reference to a fixed point of the substance, a

¹⁾ l.c., p. 211. As I shall not consider the magnetic rotation, I have put $\epsilon_1 = 0$.

definite value τ , the same for the two kinds of rays. If then, V_1' und V_2' are the velocities of propagation, taken relatively to the moving ponderable matter, we shall have

$$\omega = \frac{\pi}{\tau} \left(\frac{1}{V_1'} - \frac{1}{V_2'} \right). \dots \dots \dots (11)$$

For the velocity of one of the circularly polarized rays, LARMOR finds (p. 214)

$$V_1' = \frac{c}{K_1^{1/2}} - \frac{v}{K_1}, \dots \dots \dots (12)$$

where

$$K_1 = K + \frac{2 \pi \epsilon_2}{\lambda}, \dots \dots \dots (13)$$

λ being the wave-length. By substituting this value in (12), putting at the same time

$$\lambda = V_1' \tau, \dots \dots \dots (14)$$

we might obtain an equation, by means of which V_1' could be determined in function of τ . We may however simplify by observing that ϵ_2 has a very small value and that in (13) λ occurs only in a term, containing this factor ϵ_2 . For this reason, it is allowed to substitute for λ the value corresponding to $\epsilon_2 = 0$. Thus, by (12), (13) and (14)

$$\lambda = \left(\frac{c}{K^{1/2}} - \frac{v}{K} \right) \tau \dots \dots \dots (15)$$

Let us now put

$$\frac{c}{K_1^{1/2}} = U_1,$$

i. e., on account of (13), if we neglect the square of ϵ_2 ,

$$U_1 = \frac{c}{K^{1/2}} \left(1 - \frac{\pi \epsilon_2}{K \lambda} \right); \dots \dots \dots (16)$$

then (12) takes the form

$$V_1' = U_1 - \frac{U_1^2}{c^2} v \dots \dots \dots (17)$$

In order to obtain the velocity of the other circularly polarized ray, we have only to change the sign of ϵ_2 , so that we may write

$$V_2' = U_2 - \frac{U_2^2}{c^2} v, \dots \dots \dots (18)$$

where

$$U_2 = \frac{c}{K^{1/2}} \left(1 + \frac{\pi \epsilon_2}{K \lambda} \right) \dots \dots \dots (19)$$

It is to be remarked, that in this equation, as well as in (16), λ has the value (15).

Now, if we neglect terms containing v^2 , as we shall always do,

the formulae (17) and (18) are the same as the two first equations, given by LARMOR on p. 215. Further, it is there pointed out that in the result for the angle of rotation the quantities depending on the last terms of (17) and (18) disappear. Indeed

$$\frac{1}{V_1'} = \frac{1}{U_1} \left(1 + \frac{U_1}{c^2} v \right) = \frac{1}{U_1} + \frac{v}{c^2},$$

$$\frac{1}{V_2'} = \frac{1}{U_2} + \frac{v}{c^2},$$

whence

$$\omega = \frac{\pi}{\tau} \left(\frac{1}{U_1} - \frac{1}{U_2} \right).$$

So far, I agree with LARMOR's calculation. But, in coming to his conclusion, he has overlooked that the value of ω still contains the velocity of translation. This is seen by referring to (16) and (19). Using these, we find

$$\frac{1}{U_1} = \frac{K^{1/2}}{c} \left(1 + \frac{\pi \epsilon_2}{K \lambda} \right), \quad \frac{1}{U_2} = \frac{K^{1/2}}{c} \left(1 - \frac{\pi \epsilon_2}{K \lambda} \right),$$

$$\frac{1}{U_1} - \frac{1}{U_2} = \frac{2 \pi \epsilon_2}{K^{1/2} c \lambda},$$

and, taking from (15)

$$\frac{1}{\lambda} = \frac{K^{1/2}}{c \tau} \left(1 + \frac{v}{c K^{1/2}} \right),$$

$$\omega = \frac{2 \pi^2 \epsilon_2}{c^2 \tau^2} \left(1 + \frac{v}{c K^{1/2}} \right) \dots \dots \dots (20)$$

If the body were at rest, the velocities of the circularly polarized rays would be $\frac{c}{K_1^{1/2}}$ and $\frac{c}{K_2^{1/2}}$, if $K_2 = K - \frac{2 \pi \epsilon_2}{\lambda}$. The mean of these values, up to the first power of ϵ_2 , is

$$W = \frac{c}{K^{1/2}}.$$

If we also take into account the relation (9) and the value

$$n' = \frac{2 \pi}{\tau}$$

of the frequency, we find that (20) does not differ from my result, expressed in the equation (10).

§ 7. In order to show that the rotation must be independent of the motion of the earth, LARMOR adduces also the general considerations that are to be found in Chapter X of his work; from these the proposition may really be inferred, though not without an auxiliary hypothesis. As is well known, the theory of optical phenomena in moving bodies is simplified very much by the introduction, instead

of the time t , of the so-called "local" time t' as an independent variable, the equation

$$t' = t - \frac{1}{c^2} (v_x x + v_y y + v_z z)$$

serving to define this quantity in terms of t and the coordinates x, y, z with respect to axes fixed in the body. By means of this contrivance the electric force, exerted by a small electrically polarized particle P on an electron Q , situated at some distance, is made to be determined by equations of the same form, whether there be or not a common translation of P and Q .

Let¹⁾ m be the electric moment, varying with the time, of P , x, y, z the coordinates of the kind just mentioned in the surrounding field, r the distance to P ; then for any point in the field, at its own local time t' , the components of the said electric force will be

$$c^2 \left\{ \frac{\partial^2}{\partial x \partial y} \left(\frac{m_y}{r} \right) + \frac{\partial^2}{\partial x \partial z} \left(\frac{m_z}{r} \right) - \frac{\partial^2}{\partial y^2} \left(\frac{m_x}{r} \right) - \frac{\partial^2}{\partial z^2} \left(\frac{m_x}{r} \right) \right\},$$

etc.,

provided we take for m_x, m_y, m_z the values corresponding to the instant at which the local time in P is $t' - \frac{r}{c}$, so that the numerators in the expressions $\frac{m_x}{r}, \frac{m_y}{r}, \frac{m_z}{r}$ depend on t', x, y, z . The differentiations must be performed for a constant t' .

§ 8. We shall now suppose that a dielectric contains a very large number of particles, in which electric moments m can be excited, that the sole interaction between these consists in the above mentioned electric forces, and that for each particle the connexion between its moment and the electric force is not altered by a translation. If then, in the absence of such a motion, m_x, m_y, m_z for the different particles of the body can be certain functions of the time t , we shall obtain a state that is possible in the moving body, by supposing these moments to be exactly the same functions of the local time t' . This follows at once from what has been said in the last §. It is also easily seen that in a point fixed to the ponderable matter, the time of vibration will be the same in the two states, and that, if the first of these states consists in a propagation of light with rotation of the plane of polarization, we shall have in the second state a similar propagation, the angle between the vibrations in any

¹⁾ See my „Versuch u. s. w.“, § 33.

two points of the body being the same in the two cases. The rotation would therefore be independent of the translation, always provided we compare cases in which the frequency in a point of the body has a definite value.

§ 9. What precedes calls forth two questions. In the first place : can a substance, like the one we have supposed, really have the rotatory property? And, secondly, if this be so, is the picture we have formed of the substance, the only one that agrees with the phenomena, or are there others, equally satisfying?

The answer to the first question must undoubtedly be affirmative. Within the limits of the hypotheses of § 8 there is room for a large variety of optical properties, which may depend either on the form of the connexion between the electric force and the moment of a single particle, or on the relative position of the different particles, and a peculiar arrangement may very well produce a rotation of the plane of polarization. For this it is only necessary that the structure of the system should be asymmetric, i.e. that the system should not be in every respect equal to its reflected image. If, in such a case, we consider the electric interaction between neighbouring particles, we shall have to introduce into the equations certain terms of a rotational character. As a simple example of the required structure we may take a molecule containing 4 unequal particles situated at the angles of an asymmetric tetrahedron, and each of which may be electrically polarized.

As to the second question, it is clear that in real bodies there may very well be circumstances, differing from those we have supposed in § 8. We may e. g. conceive a movable electron, situated at one angle of the asymmetric tetrahedron, to be subject not only to the electric action of a moment, situated at one of the other angles, but also to a force of some other kind ("molecular" force), issuing from that angle. If, in such a case, the action between two elements of matter A and B were such that the action on A at the local time t' were determined by the state of B at the same local time, what has been said about two corresponding states might still be true. But this need no longer be so, if the action on A at the time t depends on the state of B at that same instant.

However this may be, it must certainly be deemed possible that after all the rotation is not altered by a uniform motion of the active substance; this possibility would however be excluded if we began by omitting in the equation (1) the term with k .

§ 10. The necessity of retaining this term may also be seen in the following way. In the fundamental equations (4) the coordinates are already taken relative to axes, moving with the medium, but the local time has not yet been introduced. We shall now do this, so that our independent variables become x, y, z and t' . We shall distinguish by accents the differential coefficients with respect to x, y, z , for a constant t' from the corresponding differential coefficients, taken for a constant t . We shall likewise denote by Div' and Rot' operations, in which the new differentiations occur in the same way as the original ones in the operations, represented by Div and Rot .

The formulae of transformation are

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right)' - \frac{v_x}{c^2} \frac{\partial}{\partial t'}, \text{ etc.}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'},$$

and, if \mathfrak{U} be any vector,

$$Rot \mathfrak{U} = Rot' \mathfrak{U} + \frac{1}{c^2} [\dot{\mathfrak{U}} \cdot \mathfrak{p}] \dots \dots \dots (21)$$

Using these, and introducing instead of \mathfrak{D} the new vector

$$\mathfrak{D}' = \mathfrak{D} + \frac{1}{4\pi c^2} [\mathfrak{p} \cdot \dot{\mathfrak{H}}], \dots \dots \dots (22)$$

we may write for the first four of the equations (4)

$$Div' \mathfrak{D}' = 0,$$

$$Div' \mathfrak{H}' = 0,$$

$$Rot' \mathfrak{H}' = 4\pi \mathfrak{D}',$$

$$Rot' \mathfrak{E} = -\dot{\mathfrak{H}}'.$$

These formulae have the same form as those which, for a body at rest, determine $\mathfrak{E}, \mathfrak{D}$ and \mathfrak{H} , as functions of x, y, z and t ; the rotation of the plane of polarization will therefore be independent of the translation, if the connexion between \mathfrak{D}' and \mathfrak{E} in one, and that between \mathfrak{D} and \mathfrak{E} in the other case correspond to each other in the same way. Now, if, according to (7), we put for the body at rest

$$\mathfrak{M} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma^2} Rot \mathfrak{E},$$

or

$$\mathfrak{D} - \frac{1}{4\pi c^2} \mathfrak{E} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma} Rot \mathfrak{E},$$

the said agreement requires for the moving system

$$\mathfrak{D}' - \frac{1}{4\pi c^2} \mathfrak{E} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma^2} Rot' \mathfrak{E}.$$

But, for this system, by (22), joined to the 5th and 7th of the equations (4),

$$\mathfrak{D}' - \frac{1}{4\pi c^2} \mathfrak{E} = \mathfrak{M};$$

so that we find by using (21)

$$\mathfrak{M} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma^2} \text{Rot } \mathfrak{E} + \frac{j}{c^2 \sigma^2} [\dot{\mathfrak{E}} \cdot \mathfrak{p}].$$

This is precisely the formula (7), if for k we take the value (3).

Physics. — H. A. LORENTZ. "*The intensity of radiation and the motion of the earth*".

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Many years ago FIZEAU ¹⁾ remarked that, if the aether does not follow the earth in its annual motion, the radiation, emitted by a terrestrial source of light or heat L , might possibly have unequal intensities in different directions. Let A be a point that is likewise fixed to the earth, and whose distance from L we shall denote by l . Then, if LA have the direction of the earth's velocity v , a vibration produced by L will have to travel over a length

$$l \cdot \frac{c}{c - v}$$

(c velocity of light), before it reaches A . On the contrary, its course will be

$$l \cdot \frac{c}{c + v},$$

if LA has the opposite direction. FIZEAU expected that the intensities received by A in the two cases would be inversely as the squares of these expressions, so that there would be a difference which one might hope to detect by means of suitable experiments with a thermo-electric battery.

From our present views regarding electric and optical phenomena in moving bodies it may be inferred that the experiment, proposed by FIZEAU would have a negative result, the amount of heat which is imparted to an absorbing body being independent of the earth's motion.

It will suffice to consider a simple case, omitting all terms depending

¹⁾ Pogg. Ann., Bd. 92, p. 652, 1854.