Citation:

H.A. Lorentz, Some considerations on the principles of dynamics in connextion with Hertz's Prinzipien der Mechnanik, in:
KNAW, Proceedings, 4, 1901-1902, Amsterdam, 1902, pp. 713-732

4. Threshold value and refractory period are complex quantities which originate in the imperfect isolation of the reflex-arc from the surrounding medium and in the passive resistances of the chemical system.

5· Augmentation and summation of the effect of stimulation are the consequence of not compensated changes (in the sense of CLAUSIUS).

6. The form which expresses the law of WEBER-FECHNER is a formula of interpolation deduced from the principle of entropy.

Dynamics. — H. A. LORENTZ. "*Some considerations on the principles of dynamics, in connexion with* HERTZ'S "*Prinzipien der Mechanik*".

In his last work HERTZ has founded the whole science of dynamics on a single fundamental principle, which by the simplicity of its form recalls NEWTON'S first law of motion, being expressed in the words that a material system moves with constant velocity in a path of least curvature ("geradeste Bahn"). By means of the hypothesis that in many cases the bodies whose motion is studied are connected to an invisible material system, moving with them, and by the aid of a terminology akin to that of more-dimensional geometry, HERTZ was able to show that all natural motions that may be described by the rules of dynamics in their usual form, may be made to fall under his law.

From a physical point of view it is of the utmost interest to examine in how far the hypothesis of a hidden system, connected with the visible and tangible bodies, leads to a clear and satisfactory view of natural phenomena, a question which demands scrupulous examination and on which physicists may in many cases disagree. On the contrary, it seems hardly possible to doubt the great advantage in conciseness and clearness of expression that is gained by the mathematical form HERTZ has chosen for his statements. I have therefore thought it advisable to consider in how far these advantages still exist, if, leaving aside the hypothesis of hidden motions, and without departing from the general use in dynamical investigations, one considers the motion of a system as governed by "forces" in the usual sense of the word.

In what follows there is much that may also be found in the book of HERTZ. This seemed necessary in order to present the subject in a connected form.

As to the authors who have, before HERTZ, published similar investigations, I need only mention BELTRAMI, LIPSCHITZ and DARBOUX.

47*

§ 1. We shall consider a system, consisting of n material points and we shall determine its position by the rectangular coordinates of all these points. The coordinates of the first point will be represented by x_1, x_2, x_3, those of the second point by x_4, x_5, x_6, etc., and any one of the coordinates by x_ν, the index ν varying from 1 to $3n$. We shall write m for the mass of the system, and m_ν for that of the point to which the index ν belongs. This implies that any one of these quantities m_ν has the same meaning as two other ones.

§ 2. We shall determine an infinitely small *displacement* of the system by the increments dx_ν (or, as we shall write in some cases, δx_ν) of the several rectangular coordinates. We shall ascribe to such a displacement a definite *length*, to be denoted by ds, and defined as the positive value that satisfies the equation

$$m\, d\, s^2 = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, dx_\nu^2 \quad \ldots\ldots\ldots \quad (1)$$

The displacement of the system may be considered as the *complex* of the displacements of the individual points, and the rectangular components of these last displacements, i. e. the differentials dx_ν, may be called the *elements* of the displacement of the system. We shall also call ds the *distance* between the positions of the system before and after the infinitely small displacement.

§ 3. Let P, P', P'' be three positions, infinitely near each other, ds, ds', ds'' the lengths of the displacements $P \rightarrow P'$, $P \rightarrow P''$, $P' \rightarrow P''$. It may be shown by (1) that any of these lengths can never be greater than the sum of the other two, so that we may construct a triangle, having ds, ds', ds'' for its sides. By *the angle between the displacements* $P \rightarrow P'$ and $P \rightarrow P''$ we shall understand the angle between the sides ds and ds' of this triangle. If we denote it by (s, s'), the elements of the first displacement by dx_ν and those of the second b dx'_ν, we shall have

$$m\, ds\, ds' \cos{(s, s')} = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, dx_\nu\, dx_\nu' \quad \ldots\ldots \quad (2)$$

In special cases the angles of the triangle may be on a straight line, so that $(s, s') = 0$ or 180°.

The above may be extended to two displacements, having the elements dx_ν and dx'_ν, the lengths ds and ds', whose initial positions do not coincide. In this case, just like in the former one, we calculate the angle between the displacements by the formula (2).

§ 4. If we have to do with a set of vector-quantities of one kind or another — but all of the same kind — each belonging to one of the material points, we shall call the complex of all these quantities a *vector in the system* or simply a *vector*. The rectangular components of the several vector quantities will be called the *elements of the vector in the system.*

From this it follows that an infinitely small displacement is itself a vector in the system, and that any vector may be geometrically represented on an infinitely small scale by such a displacement. The *length* or *value of a vector* and the *angle between two vectors* may be defined in a similar way as the corresponding quantities in the case of infinitely small displacements.

We shall often denote a vector by the letter $\mathfrak{S}$, its value by S, its elements by X_ν. Accents or other suffixes will serve to distinguish one vector from another. Other Gothic letters for vectors, and the corresponding Latin ones for their values will likewise be used. If an infinitely small displacement is to be regarded as a vector, we shall denote it by $d\mathfrak{s}$ or $\delta\mathfrak{s}$.

The value S of a vector, considered in most cases as a positive quantity, is given by the formula

$$m\,S^2 = \sum_1^{3n} {}_\nu\, m_\nu\, X_\nu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the angle $(\mathfrak{S}, \mathfrak{S}')$ between two vectors by

$$m\,S\,S'\cos(\mathfrak{S}, \mathfrak{S}') = \sum_1^{3n} {}_\nu\, m_\nu\, X_\nu\, X_\nu' . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $(\mathfrak{S}, \mathfrak{S}') = 0$, the vectors are said to have the same direction. For this it is necessary and sufficient that the ratios between the elements X_ν should be the same as those between the elements X_ν'. The ratios between the elements and the length will then likewise be the same for the two vectors. It is natural to call these last ratios the *direction-constants*. If these are α_ν, so that

$$\alpha_\nu = \frac{X_\nu}{S},$$

the equation (4) becomes

$$m\cos(\mathfrak{S}, \mathfrak{S}') = \sum_1^{3n} {}_\nu\, m_\nu\, \alpha_\nu\, \alpha_\nu' . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The angle between two vectors depends therefore on their direction-constants, or, as we may say, on their directions.

The direction-constants of a vector may not be chosen independently from each another, the relation

$$\sum_{1}^{3n} m_\nu \alpha_\nu^2 = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

having always to be satisfied.

Two vectors are said to be perpendicular to each other, if $(\mathfrak{S}, \mathfrak{S}') = 90°$.

If the angle is 180°, the vectors have opposite directions. This will sometimes be expressed by saying that the two have the same direction-constants, but that one value is positive and the other negative.

§ 5. Multiplying a vector $\mathfrak{S}$ by a positive or negative number k means, that each element is multiplied by k, and that the products are taken as the elements of a new vector, which we shall indicate by $k\mathfrak{S}$.

Two vectors $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are said to be compounded with each other, if any two corresponding elements are added algebraically, and the sums thus obtained are taken as the elements of a new vector. This is called the *resultant* or the *sum of the two vectors*, and represented by $\mathfrak{S}_1 + \mathfrak{S}_2$; it may again be decomposed into the *components* $\mathfrak{S}_1$ and $\mathfrak{S}_2$.

There are a number of theorems, closely corresponding to those in the theory of ordinary vectors. We need only mention some of them.

If $$\mathfrak{S}_1 + \mathfrak{S}_2 = \mathfrak{S}_3 \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and if h be an arbitrarily chosen direction in the system, i. e. the direction of some vector in the system, we shall have

$$S_1 \cos(\mathfrak{S}_1 , h) + S_2 \cos(\mathfrak{S}_2 , h) = S_3 \cos(\mathfrak{S}_3 , h).$$

From this it appears that, as soon as two of the vectors $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ are perpendicular to the direction h, the third will likewise be so.

It may further be shown that a given vector $\mathfrak{S}$ may always be decomposed into one component having a given direction h (or precisely the opposite direction) and a second component, perpendicular to h. This decomposition can be effected in only one way, the value of the first component being $S \cos(\mathfrak{S} , h)$. This may be positive or negative; in one case the component has the direction h, in the other it has the opposite direction.

The value of the component along h is also called the *projection* of $\mathfrak{S}$ on the direction h.

By the *scalar product* of the vectors $\mathfrak{S}_1$ and $\mathfrak{S}_2$ we understand the expression

$$S_1 S_2 \cos(\mathfrak{S}_1, \mathfrak{S}_2) \, ,$$

for which the sign $(\mathfrak{S}_1 . \mathfrak{S}_2)$ will be used.

It is also to be remarked that, in the case of (7),

$$S_3^2 = S_1^2 + S_2^2 + 2(\mathfrak{S}_1 . \mathfrak{S}_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and that we may regard the formula

$$\mathfrak{S}_1 = \mathfrak{S}_3 - \mathfrak{S}_2 ,$$

as expressing the same relation as (7). In this way the *difference* of two vectors is defined.

We shall speak of the sum and the difference of two vectors not only if these relate to the same position of the system, but likewise if they are given for different positions.

§ 6. The material points of the system are said to be *connected* with one another, if the system is, by its nature, only capable of such infinitely small displacements as satisfy certain conditions. We shall suppose that these may be expressed by i equations of the form

$$\sum_{1}^{3n}{}_{\nu}\, x_{\iota\nu}\, dx_\nu = 0, \ (\iota = 1, 2, \ldots . i), \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which the coefficients $x_{\iota\nu}$ are functions of the rectangular coordinates, but do not explicitly contain the time. Displacements agreeing with (9) are called *possible* displacements; displacements which violate the conditions are however equally *imaginable.*

A position of the system and a vector in it being given, we may examine if the vector have or not the direction of a possible displacement. If will have such a direction if its elements or its direction-constants obey i equations, similar to (9).

If two of the three vectors in (7) have the direction of a possible displacement, the third will have the same property.

§ 7. There are *directions perpendicular to all possible displacements.* If a vector $\mathfrak{S}$ is to have such a direction, it must be possible to express its elements X_ν in i quantities Ξ_ι by means of the equations

$$m_\nu X_\nu = \sum_{1}^{i}{}_{\iota}\, x_{\iota\nu}\, \Xi_\iota \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Any system of values for Ξ_ι will give a vector that has the property in question.

If, among the vectors occurring in (7), there are two that are perpendicular to all possible displacements, the same will be the case with the third vector.

A given vector may be decomposed into two components of which the one is perpendicular to all possible displacements, and the other has the direction of a possible displacement. There is only one such decomposition.

In order to show this, we may regard as unknown quantities the $3n$ elements of the second component and i quantities Ξ_ι, in which, by (10), the elements of the first component may be expressed. There is an equal number of linear equations, i of them expressing (§ 6) that the first component has the direction of a possible displacement, and the remaining $3n$ equations, that the elements of the given vector are the sums of the corresponding elements of the two components.

§ 8. The *path* of a moving system is determined by the positions it occupies one after the other. It may be considered as a succession of infinitely small displacements, which we shall call the *elements of the path.* The *length of any part of the path* is defined as the sum

$$\int ds$$

of the lengths ds of the elements of which it consists.

The *direction of a path* in one of its positions is given by the direction of an element.

We shall always think of the system as moving along a path in a definite direction. Then the coordinates x_ν, and all other quantities that have determinate values for every position in the path, may be regarded as functions of the length s of the path, reckoned from some fixed position. Accents will serve to indicate differentiation of such quantities with respect to s.

From what has been said in § 4 it follows that the quantities x'_ν are the direction-constants of the path; they will always satisfy the relation

$$\sum_{1}^{3n}{}_\nu m_\nu x'^2_\nu = m\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

as appears from (6). Using (3), we see that a vector whose elements are x'_ν has the value 1. This vector of value 1, in the direction of the path, may be called the *direction-vector.* We shall represent it by $\mathfrak{D}$.

§ 9. We define the *curvature of a path* as the vector $\mathfrak{c}$, given by

$$\mathfrak{c} = \frac{d\mathfrak{D}}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the numerator being the difference between the vectors $\mathfrak{D}$ at the beginning and at the end of an element of the path of length ds. The elements of $\mathfrak{D}$ being x'_ν, we see at once that those of $\mathfrak{c}$ are x''_ν; accordingly, in virtue of (3), the value c of the curvature is given by

$$m\,c^2 = \sum_{1}^{3n}{}_\nu m_\nu x''^2_\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By differentiating (11) one finds

$$\sum_{1}^{3n} {}_{\nu}\, m_\nu\, x'_\nu\, x''_\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the meaning of which is that the curvature is perpendicular to the path.

Let P_1 and P_2 be two paths, having in common a position A and the direction in this position, so that the direction-vector $\mathfrak{D}$, in the position A, is likewise the same for the two paths, or

$$\mathfrak{D}_{1(a)} = \mathfrak{D}_{2(a)}.$$

Let us consider elements of the two paths, beginning in A, and of *equal* lengths ds. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the direction-vectors at the ends of these elements, the vector

$$\mathfrak{c}_r = \frac{\mathfrak{D}_1 - \mathfrak{D}_2}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

may appropriately be called the *relative curvature* of the path P_1 with respect to the path P_2. Now, we may replace the numerator in (15) by $[\mathfrak{D}_1 - \mathfrak{D}_{1(a)}] - [\mathfrak{D}_2 - \mathfrak{D}_{2(a)}]$; the relative curvature is therefore related as follows to the curvatures $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of the two paths:

$$\mathfrak{c}_r = \mathfrak{c}_1 - \mathfrak{c}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Like $\mathfrak{c}_1$ and $\mathfrak{c}_2$, the relative curvature is perpendicular to both paths.

§ 10. What has been said thus far holds for every imaginable path. We shall now consider *possible paths*, i. e. such as are composed of possible infinitely small displacements. The direction-constants of such a path satisfy the i conditions

$$\sum_{1}^{3n} {}_{\nu}\, x_{\iota\nu}\, x'_\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

as may be deduced from (9).

Let there be given a position A and a direction in this position, so that the values of x_ν and x'_ν are known, and let us seek the values of x''_ν, which make the curvature c a minimum.

In solving this problem, we have to take into account equation (14) and the conditions

$$\sum_{1}^{3n} {}_{\nu}\, x_{\iota\nu}\, x''_\nu + \sum_{1}^{3n} {}_{\nu} \sum_{1}^{3n} {}_{\mu} \frac{\partial x_{\iota\nu}}{\partial x_\mu}\, x'_\mu\, x'_\nu = 0, \quad . \quad . \quad . \quad . \quad (18)$$

which are got by differentiating (17). We may therefore write for the values of x''_ν that make (13) a minimum

$$m_\nu\, x''_\nu = \sum_{1}^{i} {}_{\iota}\, x_{\iota\nu}\, P_\iota + m_\nu\, x'_\nu\, Q\,,$$

P_ι and Q being $i + 1$ quantities whose values can be determined by means of (14) and (18). The first of these equations, combined with (17) and (11), gives $Q = 0$. The solution becomes therefore

$$m_\nu x''_\nu = \sum_1^i \iota\, x_{\iota\nu} P_\iota\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the formulae (18) will serve to determine the quantities P_ι.

A possible path which, in each of the positions belonging to it, is less curved than any other possible path of the same direction may be called a *path of least curvature*. In every position through which it passes it has the property expressed by (19) or, as we may also say, its curvature is perpendicular to all possible displacements.

A path of least curvature is determined by one position, and the direction in that position.

§ 11. We shall next consider a possible path P and the path P_o of least curvature, having in common with P one position A and the direction in that position. Let, in the position A, $\mathfrak{c}_o$ be the curvature of P_o, $x''_{\nu(o)}$ the elements of this curvature, $\mathfrak{c}$ and x''_ν the corresponding quantities for P, and let us fix our attention on the relative curvature of P, with respect to the least curved path P_o. We shall denote this relative curvature by $\mathfrak{c}_f$, and call it the *free curvature* of the possible path P. It may be shown to have the direction of a possible displacement.

Indeed, we have by definition

$$\mathfrak{c}_f = \mathfrak{c} - \mathfrak{c}_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

so that the elements of $\mathfrak{c}_f$ are $x''_\nu - x''_{\nu(o)}$. Now, if we write down two times the equations (18), first for P_o and then for P, we find by subtraction

$$\sum_1^{3n} \nu\, x_{\iota\nu} \left[x''_\nu - x''_{\nu(o)}\right] = 0,$$

which proves the proposition. We may add that $\mathfrak{c}_o$ is perpendicular to $\mathfrak{c}_f$, being perpendicular to all possible displacements, and that therefore by (8)

$$c^2 = c_o^2 + c_f^2.$$

This confirms the inequality $c_o < c$.

It is easily seen that a possible path is wholly determined if one knows one position belonging to it, the direction in that position and the free curvature in all positions.

§ 12. Let P be a possible path. From every position A lying in it we make the system pass to a *varied position* A', by giving to it an infinitely small displacement $\delta\mathfrak{s}$, for whose elements we write δx_ν, these elements being supposed to be continuous functions of the length s of the path, reckoned along P.

The path P' that is determined by the succession of the new positions A' will be called the *varied path* and the letter δ will serve to indicate the difference between quantities relating to this path and the corresponding ones relating to the original path. We shall use the sign d, if we compare the values of some quantity at the beginning and the end of an element ds of the path P.

It is easy to obtain an expression for the variation in the length of an element. Starting from (1), we find

$$m\,\delta\,d\,s = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, \delta\, d\, x_\nu = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, d\,\delta\, x_\nu =$$

$$= d\sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, \delta\, x_\nu - d\,s.\sum_{1}^{3n}{}_{\nu} m_\nu\, x''_\nu\, \delta\, x_\nu,$$

and we may simplify this by introducing the notation for the scalar product of two vectors, and writing $(\delta\,\mathfrak{s})_s$ for the projection of $\delta\mathfrak{s}$ on the direction of the path. The sums on the right hand side may then be replaced by

$$m\,(\mathfrak{D}\,.\,\delta\,\mathfrak{s}) = m\,(\delta\,\mathfrak{s})_s$$

and

$$m\,(\mathfrak{c}\,.\,\delta\,\mathfrak{s})\,.$$

In the last expression, in virtue of (20)

$$(\mathfrak{c}\,.\,\delta\,\mathfrak{s}) = (\mathfrak{c}_o\,.\,\delta\,\mathfrak{s}) + (\mathfrak{c}_f\,.\,\delta\,\mathfrak{s})\,,$$

and this is reduced to the last term, if we confine ourselves to *possible* virtual displacements $\delta\,\mathfrak{s}$, these being perpendicular to $\mathfrak{c}_o$. Finally

$$\delta\,d\,s = d\,(\delta\,\mathfrak{s})_s - (\mathfrak{c}_f\,.\,\delta\,\mathfrak{s})\,d\,s\,, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

a result, which can be illustrated by simple geometrical examples.

§ 13. It is to be remarked that the varied path of which we have spoken in the last § is not in general a possible path. This will however be the case, if the i equations (9) admit of complete integration, i. e. if the connexions may be expressed by i equations between the coordinates.

Systems having this peculiar property are called by HERTZ *holonomic*. For these, the equation (21) gives the variation which arises if one possible path is changed into another, infinitely near it, and *likewise possible*.

If now the original path were one of least curvature, we should have $\mathfrak{c}_f = 0$, and by integration over some part of the original path, in the supposition that the initial and final positions are not varied,

$$\delta\int_1^2 d\,s = 0\,.$$

This shows that for holonomic systems the paths of least curvature are at the same time *geodetic* paths.

§ 14. In considering the motion in relation to the time t, we shall indicate differentiations with respect to this variable either by the ordinary sign or by a dot. If some quantity φ may be conceived as a function of t and likewise as one of the length of path s, we have the relation

$$\frac{d\varphi}{dt} = \frac{d\varphi}{ds} \cdot \frac{ds}{dt}, \qquad \dot{\varphi} = \varphi' \frac{ds}{dt}.$$

We shall define the *velocity* $\mathfrak{v}$ of the system as the complex of the velocities of the individual points. Its elements are $\dot{x}_\nu = x'_\nu \cdot \frac{ds}{dt}$ and the vector itself is

$$\mathfrak{v} = \frac{ds}{dt} \mathfrak{D}.$$

The *direction of the velocity* is that of the path, so that we may write

$$\mathfrak{v} = v\, \mathfrak{D} \qquad (22)$$

and the *value* is

$$v = \frac{ds}{dt}.$$

If the value is determined by (3), the *kinetic energy* is easily found to be

$$T = \tfrac{1}{2}\, m\, v^2.$$

By the *acceleration* $\mathfrak{f}$ of the system we understand the complex of the accelerations of all the material points. Thus the elements of $\mathfrak{f}$ are $\ddot{x}_\nu$, and

$$\mathfrak{f} = \dot{\mathfrak{v}},$$

An interesting result is obtained if in this equation we use (22), (12) and (20). We are then led to the following decomposition of the acceleration into three components:

$$\mathfrak{f} = v\dot{\mathfrak{D}} + \dot{v}\mathfrak{D} = v^2 \frac{d\mathfrak{D}}{d\,s} + \dot{v}\,\mathfrak{D} = v^2\, \mathfrak{c} + \dot{v}\,\mathfrak{D} = v^2\, \mathfrak{c}_o + v^2\, \mathfrak{c}_f + \dot{v}\,\mathfrak{D} \quad . \qquad (23)$$

The first component is perpendicular to all possible displacements, the second has the direction of the free curvature and the third that of the path.

It is easily seen that a possible motion will be quite determined, if we know one position, the velocity in that position and, for every instant, the second and the third component of the acceleration. Indeed, the second component determines the free curvature, and by this the change in the direction of the path, and the third component determines the change in the value of the velocity.

§ 15. Let the material points of the system be acted on by forces, in the usual sense of the word, and let X_ν be the rectangular components of these. We shall take together all these forces, so that we may speak of them all as of one thing, but in doing so we shall slightly depart from the way in which we have defined the velocity and the acceleration. We begin by multiplying each individual force by $\frac{m}{m'}$, m' being the mass of the point on which it acts, and m the mass of the whole system, and we understand by *the force $\mathfrak{F}$ acting on the system* the complex of these new vectors.

The elements of $\mathfrak{F}$ are therefore $\frac{m}{m_\nu} X_\nu$.

Assigning to $\mathfrak{F}$ a definite direction and a definite value will of course imply that all the forces acting on the material points of the system are given in direction and magnitude.

The definition of the force $\mathfrak{F}$ has been so chosen that the work of the forces in an infinitely small displacement, i. e. the expression

$$\sum_{1}^{3n} {}_\nu \, X_\nu \, d\, x_\nu,$$

becomes equal to the scalar product $(\mathfrak{F}.\, d\,\mathfrak{s})$.

§ 16. Every force $\mathfrak{F}$ may be decomposed into one component $\mathfrak{F}_0$, perpendicular to all possible displacements, a second component $\mathfrak{F}_1$, having the direction of a possible displacement and perpendicular to the path, and a third component $\mathfrak{F}_2$, in the direction of the path.

One can conduct this operation in two steps. Replace first (§ 7) $\mathfrak{F}$ by $\mathfrak{F}_0$, perpendicular to all possible displacements, and $\mathfrak{F}'$, in the direction of such a displacement. This being done, we have to decompose (§ 5) $\mathfrak{F}'$ into a force $\mathfrak{F}_2$, along the path, and a force $\mathfrak{F}_1$, perpendicular to it. The latter component will have the direction of a possible displacement, because $\mathfrak{F}'$ and $\mathfrak{F}_2$ have such directions.

For a given force the three components are wholly determinate.

§ 17. We may imagine each material point to be acted on by a force in the direction of the acceleration of the point and equal to the product of the acceleration and the mass. We shall denote by $\mathfrak{S}$ the force acting on the system in this special case, by $\mathfrak{S}_0$, $\mathfrak{S}_1$, $\mathfrak{S}_2$ its components in the above mentioned directions.

Now we have in the supposition just made $X_\nu = m_\nu \ddot{x}_\nu$, from which we find $m \ddot{x}_\nu$ for the elements of $\mathfrak{S}$, $\mathfrak{S} = m\, \mathfrak{f}$ for the force itself, and, by (23),.

$$\mathfrak{G}_0 = m v^2 \mathfrak{c}_o, \quad \mathfrak{G}_1 = m v^2 \mathfrak{c}_f, \quad \mathfrak{G}_2 = m \dot{v} \mathfrak{D}$$

for the three components.

§ 18. What precedes has prepared us for the consideration of the fundamental principles by which the motion of the system under the action of given forces is to be determined. We may in the first place start from the following assumptions:

a. The system will have the acceleration $\mathfrak{f}$, if the force is precisely $\mathfrak{G} = m\mathfrak{f}$.

b. Two forces $\mathfrak{F}_a$ and $\mathfrak{F}_b$ may have the same influence on the motion. For this it is necessary and sufficient that the force

$$\mathfrak{F}_a - \mathfrak{F}_b$$

should be perpendicular to all possible displacements.

Let the system be subject to the force $\mathfrak{F}$ with the components $\mathfrak{F}_0$, $\mathfrak{F}_1$, $\mathfrak{F}_2$, and let the acceleration be $\mathfrak{f}$. Then, by the first principle, $\mathfrak{F}$ has the same influence as $\mathfrak{G} = m\mathfrak{f}$, and by the second principle $\mathfrak{F} - \mathfrak{G}$ must be perpendicular to all possible displacements. This amounts to the same thing as $\mathfrak{F}_1 = \mathfrak{G}_1$, $\mathfrak{F}_2 = \mathfrak{G}_2$, or

$$\mathfrak{F}_1 = m v^2 \mathfrak{c}_f, \quad \mathfrak{F}_2 = m \dot{v} \mathfrak{D}. \quad \ldots \ldots \quad (24)$$

It will be immediately seen that the above assumptions are equivalent to D'ALEMBERT's principle. We might also have replaced them by the following rule:

Decompose the acceleration into two components $\mathfrak{f}_0$ and $\mathfrak{f}'$, the first perpendicular to all possible displacements, and the second in the direction of such a displacement. Decompose the force $\mathfrak{F}$ in the same way into the components $\mathfrak{F}_0$ and $\mathfrak{F}'$. Then the equation of motion will be

$$\mathfrak{F}' = m\mathfrak{f}'.$$

This leads directly to the equations (24), by which it is clearly seen that the change in direction of the path is determined by the component $\mathfrak{F}_1$, and the change in the value of the velocity by the component $\mathfrak{F}_2$. It is to be kept in mind that the first of the formulae (24) is a *vector-equation*. In general the free curvature, as well as the force $\mathfrak{F}_1$, may have different directions, in some cases a great many of them. The equation does not only show us *to what amount* the path deviates from one of least curvature, but also *to which side* the deviation takes place.

If $\mathfrak{F} = 0$, we have $\mathfrak{c}_f = 0$ and $\dot{v} = 0$; we are then led back to the fundamental law of HERTZ.

§ 19. Let us now return to the equation (21), taking for the original path one that is described under the action of the existing forces. Attending to (24), we may write in (21)

$$(\mathfrak{c}_f \, . \, \delta \, \mathfrak{s}) = \frac{1}{mv^2} (\mathfrak{F}_1 \, . \, \delta \, \mathfrak{s}) = \frac{1}{mv^2} [(\mathfrak{F} \, . \, \delta \, \mathfrak{s}) - (\mathfrak{F}_0 \, . \, \delta \, \mathfrak{s}) - (\mathfrak{F}_2 \, . \, \delta \, \mathfrak{s})].$$

Now we have $(\mathfrak{F}_0 \, . \, \delta \, \mathfrak{s}) = 0$, because $\mathfrak{F}_0$ is perpendicular to the virtual displacement. Further:

$$(\mathfrak{F}_2 \, . \, \delta \, \mathfrak{s}) = m \, \dot{v} \, (\mathfrak{D} \, . \, \delta \, \mathfrak{s}) = m \, \dot{v} \, (\delta \, \mathfrak{s})_s = m \, v' \, v \, (\delta \, \mathfrak{s})_s = m \, v \frac{dv}{ds} (\delta \, \mathfrak{s})_s,$$

so that (21) becomes

$$\delta \, d \, s = d \, (\delta \, \mathfrak{s})_s + \frac{1}{v} \, d \, v \, (\delta \, \mathfrak{s})_s - \frac{1}{mv^2} (\mathfrak{F} \, . \, \delta \, \mathfrak{s}) \, d \, s,$$

or, multiplied by $m \, v$,

$$m \, v \, \delta \, d \, s + \frac{1}{v} (\mathfrak{F} \, . \, \delta \, \mathfrak{s}) \, d \, s = m \, d \, [v \, (\delta \, \mathfrak{s})_s \,].$$

The scalar product $(\mathfrak{F} \, . \, \delta \mathfrak{s})$ on the left is the work of the force for the virtual displacement; in the case of a *conservative* system with potential energy U, it may be denoted by $- \, \delta \, U$. The result therefore takes the form

$$m \, v \, \delta \, d \, s - \frac{1}{v} \delta \, U d \, s = m \, d \, [v \, (\, \delta \, \mathfrak{s})_s \,] \, . \quad . \quad . \quad . \quad (25)$$

§ 20. Thus far, we have spoken only of a varied path, but not of a *varied motion*; we have said nothing about the instants at which we imagine the varied positions to be reached. In this respect we may make different assumptions, and among these there are two, which lead to a simple result of the equation (25), if integrated over a part of the path.

a. Let the varied positions A' be reached *at the same moments* as the corresponding positions A in the original motion. Then

$$m \, v \, \delta \, d \, s = m \, v \, \delta \, v \, . \, d \, t = \delta \, T \, d \, t \, ;$$

(25) becomes

$$\delta \, (T - U) \, d \, t = m \, d \, [v \, (\delta \, \mathfrak{s})_s] \, ,$$

and, if integrated along the path which the system travels over between the instants t_1 and t_2, in the supposition that $\delta \, \mathfrak{s} = 0$ for $t = t_1$ and $t = t_2$,

$$\delta \int (T - U) \, d \, t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This is HAMILTON's principle, which is in itself sufficient for the determination of the motion really taking place under the action of given forces, and from which we may infer e. g. that in the course of this motion

$$T + U = E$$

remains constant.

b. In the second place we shall assume that *in the varied motion the energy $T + U$ has the same constant value E as in the original motion.* This value E having been chosen, and U being known for every position, the value of the velocity is given by

$$v = \sqrt{\frac{2}{m}(E - U)}.$$

This second assumption therefore, as well as the first, leaves no doubt as to the velocity with which the system is supposed to travel along its varied path.

The total energy remaining constant, we have now

$$\delta U = -\delta T = -m v \delta v,$$

and (25) becomes

$$\delta (v\, d s) = d\,[v\,(\delta \mathfrak{s})_s],$$

or

$$\delta (\sqrt{E - U}\, d s) = \sqrt{\tfrac{1}{2} m}\, d\,[v\,(\delta \mathfrak{s})_s] \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence, if we integrate along a certain part of the path, supposing again the extreme positions to remain unchanged,

$$\delta \int \sqrt{E - U}\, d s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This is the principle of least action in the form that has been given to it by JACOBI. Indeed we may define the action along a path of the system as the integral that occurs in the equation (28) [1]. Its value may be calculated for any path $A_1 A_2$ whatsoever. For the sake of brevity we shall denote it by $V_{A_1}^{A_2}$.

Both the principle of HAMILTON and that of JACOBI have been here obtained by the consideration of the variation in the length of an element of a *curved* path that is caused by virtual displacements of the system. It is clear that both principles hold for every system, be it *holonomic*, or not, the only condition being that the virtual displacements do not violate the connexions. We must however keep in mind that it is only in the case of holonomic systems that the varied motion may be said to be, as well as the original one, a *possible* motion. Hence, if we wish to compare the motion taking place under the action of the given forces with another motion, differing infinitely little from it, and such that it is not excluded by the connexions, the two principles will only be true for holonomic

[1]) The action is usually defined as the integral, multipied by $\sqrt{2m}$.

systems [1]). The variations of the integrals occurring in (26) and (28) will be 0, if the original motion is not only a possible one, but such that it may really take place under the influence of the acting forces. We shall call such a motion a *real* or a *natural* one.

§ 21. We shall conclude by briefly showing how some well known propositions may be presented in a form, agreeing with what precedes. These propositions relate to holonomic systems. Let us therefore assume that the connexions are expressed by i equations which must be satisfied, independently of the time, by the coordinates x_ν, and let us confine ourselves to *possible positions*, i. e. such as agree with these i conditions. We might determine these positions by $3n-i$ "free" coordinates, but in what follows, it is not necessary to do so.

If, in addition to the equations expressing the connexions, we assume still one other equation between the coordinates, we shall call the totality of positions satisfying that equation a *surface of positions.*

In case one of these positions A is reached by a certain path — the other positions in this path not all of them belonging to the surface — the path may be said to *cut* the surface in the position A. For simplicity's sake it will be supposed in such a case that the surface and the path have only that one position in common.

Starting from a position A, which belongs to, or lies in a surface of positions S, we may give to the system infinitely small displacements, in such directions that by them the position does not cease to belong to the surface.

Another infinitely small possible displacement ds whose direction is perpendicular to all those *displacements in the surface* may be said to be *perpendicular to the surface.* It is easily shown that a displacement of the latter kind may always be found and that its direction is entirely determinate.

Let S be a surface of positions, A a position that does not belong to it but is infinitely near others that do, B a position in the surface, such that the infinitely small displacement $A \rightarrow B$ is perpendicular to S, and C any position in S infinitely near B. Let ϑ be the angle between the displacements $A \rightarrow B$ and $A \rightarrow C$, and let us denote by $\overline{AB}$ and $\overline{AC}$ the lengths of these displacements. Then

$$\overline{AB} = \overline{AC} \cos \vartheta.$$

[1]) See HÖLDER, Gött. Nachr., 1896, p. 122.

This follows from what has been said in § 5, if we consider that $A \rightarrow C$ is the resultant of the displacements $A \rightarrow B$ and $B \rightarrow C$.

§ 22. Henceforth we shall treat only of *natural* motions, taking place with a fixed value E of the total energy, which we choose once for all. We shall suppose that, if O and A are any two positions, there is one and only one such a natural motion which leads from O to A. The action along the path of this motion,

$$V_O^A = \int_O^A \sqrt{(E-U)}\, ds,$$

will have a definite value, depending on the coordinates of O and A, and we shall examine the variations of this action, if we change the final position A, the initial one being fixed.

In the first place it is clear that, if we move A along a path issuing from O, V_O^A will be the greater, the farther A recedes from O. Indeed, V_O^A presents a certain analogy with the length of the path, the difference being that, in calculating the action, we must multiply each element ds by the factor $\sqrt{E-U}$, which changes with the position.

The increment of the action, corresponding to an element of the path, is obviously

$$\sqrt{E-U}\, ds.$$

In the second place we compare two paths, both issuing from O, but in directions that differ infinitely little from each other. We shall proceed along these so far, say till we have reached the positions A and A', that the action is equal in the two cases, i. e.

$$V_O^A = V_O^{A'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Now, the motion $O \rightarrow A'$ may be conceived as the result of an infinitely small variation of the motion $O \rightarrow A$; we may therefore apply the equation we deduce from (27), if we integrate from O to A. On account of (29), we get 0 on the left-hand side, hence, the projection $(\delta \mathfrak{s})_s$, which vanishes for the position O, must likewise be 0 for the position A, and the infinitely small displacement $A \rightarrow A'$ is found to be perpendicular to the path OA.

We may next fix our attention on all paths that issue from a definite position O. In each of these we choose a position A at such a distance from O, that the action V_O between O and these positions has the same value for all of them. The positions A will belong to a certain surface and this will be cut under right angles by all the paths. We may therefore call these latter the orthogonal trajectories of the surfaces

$$V_O = const. \qquad (30)$$

Let S be that one of these surfaces, to which a certain position A belongs, and let B be some position, infinitely near A, and further from O than the surface S. In order to find an expression for the difference $V_O^B - V_O^A$, we consider also the surface S', likewise belonging to the group (30), and containing the position B; this surface will be cut in a certain position C by the path OA prolonged. If ϑ is the angle between $A \rightarrow B$ and the direction of the path in A, we shall have

$$V_O^B - V_O^A = V_O^C - V_O^A = \sqrt{E-U}.\overline{AC} = \sqrt{E-U}.\overline{AB}\cos\vartheta \quad . (31)$$

§ 23. Instead of considering the paths issuing from one and the same position O, we may also begin by choosing a surface of positions S_o, and think of all the motions in which the system starts from a position belonging to this surface, in a direction perpendicular to it. We shall suppose that any given position A may be reached by one and only one of these motions, and we shall write $V_{S_o}^A$ for the action along the path leading from S_o to A.

This function has properties similar to those of the function we have studied in the preceding §. The paths are the orthogonal trajectories of the surfaces

$$V_{S_o}^A = const.,$$

and the change in the action, caused by an infinitesimal variation of A is given by a formula of the same form as (31).

§ 24. The values of V_O^A and $V_{S_o}^A$ may depend in many different ways on the coordinates of the variable position A, according to the choice of the initial position O or the surface S_o from which we start. All these different functions have however one common property, which follows immediately from what has been said, and for which a concise form of expression is obtained in the following way.

If Q is a function of the coordinates, we may, for every infinitely small possible displacement ds, beginning in a position A, calculate the ratio

$$\frac{dQ}{ds} \qquad (32)$$

The value of this will of course depend on the direction of $d\mathfrak{s}$, and the position A being chosen, there will be one definite direction (perpendicular to the surface $Q=$ const.), for which the ratio takes the largest positive value. Now, if we denote this maximum value of (32) by DQ, the property in question of the functions V may be expressed by

$$DV_0 = \sqrt{E-U} \text{ and } DV_{S_0} = \sqrt{E-U}.$$

These formulae may be written in the form of a partial differential equation which is satisfied by V_0 and V_{S_0}.

§ 25. Let R be any solution of this differential equation, i. e. a function of the coordinates, such that

$$DR = \sqrt{E-U}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

then the orthogonal trajectories of the surfaces

$$R = const. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

are natural paths of the system.

The proof of this is as follows. Imagine the infinitely small displacements $d\mathfrak{s}$, lying between the two consecutive surfaces

$$R = C \text{ and } R = C + dC,$$

that is to say, the displacements whose initial position belongs to the first and whose final position belongs to the second surface, and subject them to the further condition that they are to be perpendicular to the first surface. Then we have by (33)

$$\frac{dC}{ds} = \sqrt{E-U}, \qquad \sqrt{E-U}\, ds = dC,$$

so that the action is the same for all these elements, whichever be the position in the surface from which they start.

On the contrary, if $d\mathfrak{s}'$ is another element of path between the two surfaces, not perpendicular to them, but making an angle ϑ with one of the first-named displacements $d\mathfrak{s}$ in the immediate vicinity, we shall have

$$d\,s' = \frac{d\,s}{\cos\vartheta},$$

and for the action along $d\mathfrak{s}'$

$$\frac{d\,C}{\cos\vartheta}.$$

It appears from this that

$$\delta\{\sqrt{E-U}\,d\,s\} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

if we pass from an element $d\mathfrak{s}$, perpendicular to $R=C$, to an element $d\mathfrak{s}'$, lying between the same two surfaces, and of such a direction that ϑ is infinitely small.

This being established, we may take an arbitrarily chosen part MN of an orthogonal trajectory of the surfaces (34), we may divide it into elements by means of surfaces belonging to the group, and infinitely near each other, and we may give to MN an infinitely small variation, without however changing the positions M and N. Then, applying what has just been said to every element of MN, and integrating, we find

$$\delta \int \sqrt{E-U}\, ds = 0,$$

showing that MN is a natural path.

At the same time the meaning of the function R becomes apparent. Its value in a certain position A is the action along a trajectory, ending in A and beginning at the surface

$$R = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

§ 26. We shall now assume that we know a function $R(c)$ of the coordinates, satisfying the differential equation (33) and containing an arbitrary constant c.

Then $\frac{\partial R}{\partial c}$, which is itself a function of the coordinates, will have the same value for all positions lying on a path P, perpendicular to (36).

To show this, we consider the consecutive surfaces whose equations are

$$R(c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and

$$R(c + dc) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Let A_0 be the position in the first surface where the path P begins, A any position belonging to P. We shall suppose this path A_0A to cut the surface (38) in a position A_1, which is, of course, infinitely near A_0. We shall finally think of the path, such as there certainly is one, leaving the surface (38) in a perpendicular direction and terminating in A. It has a definite initial position in (38), infinitely near A_1, and which we shall call A'.

Since

$$V_A^A = V_{A'}^A,$$

as may be deduced from (27), we have

$$V_{A'}^A - V_{A_0}^A = V_{A_1}^A - V_{A_0}^A = - V_{A_0}^{A_1} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Now, $V_{A_0}^A$ and $V_{A'}^A$ are the values of $R(c)$ and $R(c+dc)$ for the position A. Consequently, the first member of (39) is the value of $\frac{\partial R}{\partial c}\, dc$ for this position, and

$$\frac{\partial R}{\partial c} = -\frac{1}{dc} V_{A_0}^{A_1} \quad \ldots \ldots \ldots \quad (40)$$

The right-hand member of this formula remaining the same if we move the system along the path P, the proposition is proved.

§ 27. A remarkable and well known theorem of JACOBI is a direct consequence of our last proposition. If we have found a function R of the coordinates, satisfying the differential equation (33) and containing, besides an additive constant, $3n—i—1$ other arbitrary constants c_1, c_2, etc., the values of $\frac{\partial R}{\partial c_1}, \frac{\partial R}{\partial c_2}$ etc. will not change, while the system describes a path, perpendicular to the surface

$$R(c_1, c_2, c_3, \ldots) = 0. \quad \ldots \ldots \ldots \quad (41)$$

The $3n—i—1$ equations of such a path will therefore be of the form

$$\frac{\partial R}{\partial c_1} = \gamma_1, \frac{\partial R}{\partial c_2} = \gamma_2, \text{ etc.} \quad \ldots \ldots \quad (42)$$

where γ_1, γ_2, etc. are constants.

The total number of the constants c and γ is $2(3n—i—1)$; this is just sufficient for the representation of every path that may be described with the chosen value of the energy. If we confine ourselves to fixed values of the constants c, and change those of the constants γ, we shall obtain all paths that are perpendicular to one and the same surface (41). One of these paths will be distinguished from the other by the value of the action along the parts of the path, lying between the surface (41) on one, and each of the surfaces

$$R(c_1 + d\,c_1, c_2, c_3, \ldots) = 0$$
$$R(c_1, c_2 + d\,c_2, c_3, \ldots) = 0$$
$$R(c_1, c_2, c_3 + d\,c_3, \ldots) = 0,$$

on the other side. Indeed, by (42) the action along these parts is $-\gamma_1 dc_1$, $-\gamma_2 dc_2$ etc.

By giving other values to the constants c, we shall change the surface (41) and we shall find the paths that are perpendicular to the new surface.