Physics. — Dr. G. BAKKER: *"Contribution to the theory of elastic substances."*

If we leave electrical and magnetical forces out of consideration, the forces acting on a body are gravitation, external pressure or tension and the internal molecular pressure and thermic pressure. Though in the theory of elasticity the substance is substituted by a continuous agent and we have therefore strictly speaking not to deal with mutual action of molecules, I shall yet keep to the usual term, though the term cohesion seems more suitable to me than the term molecular pressure.

In the theory of elasticity, just as in the theory of capillarity, forces are assumed, which are only perceptible at exceedingly small distances. If these forces are supposed to have a potential, the potential function

$$- f \frac{e^{-qr}}{r}$$

which is a special case of the general function of Dr. C. NEUMANN,

$$\frac{A e^{-\alpha r}}{r} + \frac{B e^{-\beta r}}{r} + \frac{C e^{-\gamma r}}{r} + \cdots$$

might be of great use here, for if we take q very large, the forces between two volume-elements will rapidly decrease with the distance.

In his thermo-dynamical theory of capillarity VAN DER WAALS has found this potential function to be a probable function for the capillary forces. Afterwards I have further discussed this function in two papers, presented to the Academy the 28th of October and the 25th of November 1899, and I further applied it in my treatise "Zur Theorie der Kapillarität" (Zeitsch. für phys. Chemie XXXIII, 4. 1900).

Let us imagine an "infinitely small" volume-element in the body in consideration, and let us take that space as unity of volume. If U, B and ϑ are respectively the virial of the external forces, that of the molecular attraction and that of the thermic pressure, then the total virial per unity of volume is e.g.:

$$F = U + B + \vartheta \ ^{[1]} \quad \ldots \quad \ldots \quad (1)$$

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[1] The influence of gravitation is left out of consideration.

I imagine the element to have the shape of a cube, the sides of which are parallel to the principal pressure-axes of the point in consideration, which I take as coordinate axes. Then the general expression for the virial:

$$F = - \Sigma \tfrac{1}{2} (X x + Y y + Z z)$$

gives immediately:

$$U = \tfrac{1}{2} (p_1 + p_2 + p_3) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

p_1, p_2 and p_3 representing the principal pressures.

If S_1, S_2 and S_3 represent the molecular tensions in the same directions and θ the thermic pressure in the point in question, then:

$$p_1 = \theta - S_1, \quad p_2 = \theta - S_2 \quad \text{and} \quad p_3 = \theta - S_3$$

or

$$p_1 + p_2 + p_3 = 3\,\theta - (S_1 + S_2 + S_3).$$

So:

$$U = \frac{3}{2}\,\theta - \frac{1}{2}\,(S_1 + S_2 + S_3). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

As the thermic pressure depends only on the condition of the substance in the immediate neighbourhood of the point [1]), where the value of the virial of the thermic pressure is θ [2]), we may take for θ also the value which this quantity would have if *at the same temperature* the substance round the point in consideration had the same density as in the point itself. If we take for the agent, which in these considerations is used as a substitute for the body which is thought to be isotropical, the potential function:

$$-f \frac{e^{-qr}}{r}$$

the following differential equation holds good for that agent:

$$\nabla_2\, V = q^2\, V + 4\pi f\varrho \ \ [3]).$$

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[1]) In contrast to the molecular forces of attraction.

[2]) Zeitschrift für phys. Chemie XXXIII, 4 1900 p. 478.

[3]) Konink. Akad. v. Wetenschappen: Proc. Nov. 25th 1899 p. 2.

For a region which is large enough and for which we think the density ϱ to be the same everywhere, $\nabla_2 V = 0$ and so:

$$V = - 4\pi f \lambda^2 \varrho$$

or substituting a for $2 \pi f \lambda^2$: $\qquad\qquad \left(\lambda = \dfrac{1}{q}\right)$

$$V = - 2a\varrho$$

the tension S now becoming:

$$S = \frac{V^2}{4\pi f \lambda^2} = a\varrho^2 \ \ ^1) \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

If in this case we call the pressure p, then:

$$p = \theta - a\varrho^2 \ \ ^2) \ . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$

p is the pressure of the homogeneous phase with the density ϱ of the point in question and at the temperature of that point.

From (5) follows:

$$\frac{3}{2} p v = \frac{3}{2} \theta v - \frac{3}{2} a \varrho \ \ (v = \text{specific volume}) \ . \ . \ . \quad (6)$$

If F_1 is the virial of the homogeneous mass per unity of mass, then:

$$F_1 = \frac{3}{2} p v + \frac{3}{2} a \varrho + \vartheta_1 \ . \ . \ . \ . \ . \ . \quad (7)$$

for, as may be easily shown, the virial is $\dfrac{3}{2} \times$ the potential energy with reversed sign. (See Zeitschrift für phys. Chemie XXI. 3. 1896,

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[1] Konink. Akad. v. Wetenschappen: Proc. Nov. 25th 1899 p. 219 and 320.

[2] If the expression $\theta = \dfrac{RT}{v - b}$ held for the thermic pressure for an isotropical substance with a certain density, we should get the same equation of state as that of VAN DER WAALS' for gases and liquids.

pag. 503). Further $\dfrac{3}{2} p v$ is the virial of the external forces and ϑ_1 that of the thermic pressure (per unity of mass).

If we substitute in (7) the value for $\dfrac{3}{2} pv$ derived from (6) we get:

$$F_1 = \frac{3}{2} \, \theta \, v + \vartheta_1 \, .$$

If we suppose that the total virial of the mass-unity is a pure function of the temperature, just as for liquids and gases, and further that ϑ_1 depends only on the density (and temperature) then :

$$F_1 \varrho = F \text{ and } \vartheta_1 \varrho = \vartheta$$

and so according to (1):

$$U + B + \vartheta = \frac{3}{2} \theta + \vartheta$$

By substitution in (3):

$$B = \frac{1}{2} \, (S_1 + S_2 + S_3) \; . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or in words :

the virial of the molecular forces per unity of volume is half the sum of the three principal tensions.

If we put $A = -f$ and $B = 0$ in the expressions which in the paper already mentioned I found for the tensions p_{xx}, p_{yy} and p_{zz} (pag. 318), through which the potential function

$$\frac{Ae^{-qr} + Be^{qr}}{r} \quad \text{becomes} \quad -f\frac{e^{-qr}}{r}$$

we get:

$$- 8 \, \pi \, f \, p_{xx} = \left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dy}\right)^2 - \left(\frac{dV}{dz}\right)^2 - q^2 \, V$$

$$- 8 \, \pi \, f \, p_{yy} = \left(\frac{dV}{dy}\right)^2 - \left(\frac{dV}{dz}\right)^2 - \left(\frac{dV}{dx}\right)^2 - q^2 \, V$$

$$- 8 \, \pi \, f \, p_{zz} = \left(\frac{dV}{dz}\right)^2 - \left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dy}\right)^2 - q^2 \, V$$

By addition:

$$- 8 \pi f (p_{xx} + p_{yy} + p_{zz}) = - \left\{ \left(\frac{dV}{dx} \right)^2 + \left(\frac{dV}{dy} \right)^2 + \left(\frac{dV}{dz} \right)^2 \right\} - 3 q^2 V$$

The sum of the tensions $p_{xx} + p_{yy} + p_{zz}$ is therefore independent of the direction of the sides of the cube-shaped element in consideration. We may therefore represent the sum by $S_1 + S_2 + S_3$ and find then:

$$B = \frac{R^2}{16 \pi f} + \frac{3 V^2}{16 \pi f \lambda^2} \quad \cdots \cdots \quad (9) \left(\lambda = \frac{1}{q} \right)$$

The tensions S_2 and S_3 normal to the lines of force appeared to be the same. We found for them:

$$S_2 = S_3 = \frac{R^2}{8 \pi f} + \frac{V^2}{8 \pi f \lambda^2}$$

and

$$S_1 = \frac{R^2}{8 \pi f} - \frac{V^2}{8 \pi f \lambda^2}$$

while we found for the potential energy per unity of volume:

$$W = - \frac{R^2}{8 \pi f} - \frac{V^2}{8 \pi f \lambda^2} .$$

Now we can easily derive the relation:

$$S_2 - S_1 = - 3 W - 2 B \quad \cdots \cdots \quad (10)$$

or in words:

The difference of the tensions normal to and in the direction of the lines of force (per unity of surface) *is three times the work required to rarefy the substance infinitely, diminished with twice the virial of the molecular forces of attraction* (per unity of volume).

Dilatation.

If λ and μ are two constants, we have according to KIRCHHOFF for the projections of the displacements of a point, whose coordinates are x, y and z:

$$(\lambda + 2\,\mu)\frac{\partial u}{\partial x} \qquad + \lambda\frac{\partial v}{\partial y} \qquad + \lambda\frac{\partial w}{\partial z} = - X_x \quad \begin{array}{l}(- X_x \text{ is there-}\\ \text{fore a tension)}\end{array}$$

$$\lambda\frac{\partial u}{\partial x} + (\lambda + 2\,\mu)\frac{\partial v}{\partial y} \qquad + \lambda\frac{\partial w}{\partial z} = - Y_y$$

$$\lambda\frac{\partial u}{\partial x} \qquad + \lambda\frac{\partial v}{\partial y} + (\lambda + 2\,\mu)\frac{\partial w}{\partial z} = - Z_z$$

By composition:

$$(3\,\lambda + 2\,\mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = - (X_x + Y_y + Z_z).$$

The second factor in the left-side member represents the dilatation, and is generally indicated by the symbol ∇. Therefore:

$$\nabla = - \frac{X_x + Y_y + Z_z}{3\,\lambda + 2\,\mu} \quad \cdots \cdots \quad (11)$$

The quantities X_x, Y_y and Z_z correspond to the quantity which we should call the hydrostatic pressure in case of a liquid. If therefore the molecular tensions are represented by S_{xx} etc., we have:

$$X_x = 0 - S_{xx}$$

$$Y_y = 0 - S_{yy}$$

$$Z_z = 0 - S_{zz}$$

By substitution of 0 from (3)

$$X_x + Y_y + Z_z = 3\,0 - (S_{xx} + S_{yy} + S_{zz}) = 3\,0 - (S_1 + S_2 + S_3) = 2\,U$$

or according to (11)

$$\nabla = - \frac{2\,U}{3\,\lambda + 2\,\mu} \quad \cdots \cdots \cdots \quad (12)$$

Applications.

1. *Elongation of a prism.* We imagine at the two ends a force S per unity of surface in the direction of the longitudinal axis. If we apply formula (12) to every volume-element of the prism, the total increase of volume becomes:

$$\int \nabla \, d\tau = -\frac{2\int U \, d\tau}{3\,\lambda + 2\,\mu}.$$

For space-elements which are quite inclosed by others, the increase of volume for the external virial (external with regard to such an element) is neutralized by that of the surrounding ones and finally the total external virial is that of the external forces acting on the prism. That virial is in this case: $-\frac{1}{2}\,S\,l\,d$.

Therefore:

$$\int \nabla \, d\tau = -\frac{2\,U}{3\,\lambda + 2\,\mu} = \frac{S\,l\,d}{3\,\lambda + 2\,\mu} \qquad (d = \text{section}).$$

The dilatation is therefore:

$$\frac{S}{3\,\lambda + 2\,\mu}$$

2. *Dilatation of a hollow cylinder.* Let S be the corresponding force just mentioned, P and p the forces per unity of surface, normal to the outer and inner surface, taken positively in the direction of the radius, then, if l represents the length, r and R the radii internal and external, the virial of S is:

$$-\tfrac{1}{2}\,S\,l\,(\pi\,R^2 - \pi\,r^2),$$

the virial of P:

$$-\tfrac{1}{2}\,\Sigma\,P\,R = -\tfrac{1}{2}\,R \times 2\,\pi\,R\,l\,P$$

and that of p:

$$-\tfrac{1}{2}\,r \times 2\,\pi\,r\,l\,p.$$

Therefore:

$$- 2\,U = S\,l\,\pi\,(R^2 - r^2) + 2\,\pi\,l\,(P\,R^2 + p\,r^2).$$

The total increase of volume divided by the volume $\pi\,(R^2 - r^2)\,l$ yields for the dilatation:

$$\nabla = \frac{1}{3\,\lambda + 2\,\mu}\left\{S + 2\,\frac{P\,R^2 + p\,r^2}{R^2 - r^2}\right\}$$

3. *Dilatation of a spherical shell.*

Let p be the internal, P the external pressure, both calculated positively in the direction of the radii r and R, then is, according to the general expression of the external virial $-\frac{3}{2}\,pv$, the virial of p:

$$-\frac{3}{2}\,p \times \frac{4}{3}\,\pi\,r^3 \text{ and of } p: -\frac{3}{2}\,P \times \frac{4}{3}\,\pi\,R^3.$$

After having divided by $3\,\lambda + 2\,\mu$ and by the original volume $\frac{4}{3}\,\pi\,(R^3 - r^3)$, we get for the dilatation:

$$\nabla = -\frac{2\,U}{3\,\lambda + 2\,\mu} : \frac{4}{3}\,\pi\,(R^3 - r^3) = \frac{3}{3\,\lambda + 2\,\mu}\,\frac{PR^3 + pr^3}{R^3 - r^3}$$

4. OERSTED's *Piëzometer*.

If V is the external and v the internal volume, then the virial of the external pressure is $p : \frac{3}{2}\,pV$ and that of the internal pressure: $-\frac{3}{2}\,pv$. As the original volume of the substance forming the shell is $V - v$, the dilatation becomes:

$$\nabla = -\frac{1}{3\,\lambda + 2\,\mu}\,\frac{2 \times \frac{3}{2}\,(V - v)}{V - v} = -\frac{3\,p}{3\,\lambda + 2\,\mu}$$

So we see that the value of the ratio ∇ does not depend on the external or internal volume nor on the form. The external volume is therefore compressed in a proportion as if the vessel were massive, which corresponds with the views of COLLADON and STURM and is opposed to those of OERSTED.