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Physics. — J. C. SCHALKWIJK: "*Precise isothermals. I. Measurements and calculations on the corrections of the mercury meniscus with standard gas-manometers*" (*Continued.*) (Communication N°. 67 from the Physical Laboratory at Leiden, by Prof. H. KAMERLINGH ONNES).

§ 6. We now can change the formulae found so, that they represent the surface of interpolation meant in § 3 for the mean height up to the limits $R = 0$ and $\delta = 0$.

For the narrow tubes we find then:

$$f = \frac{1}{2}\,\delta R + \frac{1}{6}\,\delta^3 R + \frac{1}{96}\frac{s}{H}\,\delta R^3 (1 + \delta^2)^3 \quad . \quad . \quad . \quad \text{(I)}$$

and for small values of δ:

$$f = \delta R \left\{ 1 - \frac{\Sigma \dfrac{1}{(n!)^2 (n+1)} \left(\dfrac{s}{4H} R^2\right)^{n-1}}{\Sigma \dfrac{1}{(n!)^2} \left(\dfrac{s}{4H} R^2\right)^{n-1}} \right\} \quad . \quad . \quad . \quad \text{(II)}$$

It should be noted that in both the expressions the factor of δR is greater than $\frac{1}{2}$.

In order to be able to calculate f in the limiting cases by means of these formulae, we must introduce the value of $\frac{s}{H}$. This is not exactly known to us. Fortunately an uncertainty in $\frac{s}{H}$ is of little interest for the correspondence meant in § 4, since for small values of R, $\frac{s}{H}$ in the formula (I) occurs only in that term in which also R^3 appears, so that a change in $\frac{s}{H}$ has only little influence on f. In the same way in small values of δ the influence of a change in the value of $\frac{s}{H}$ is unimportant for values of R smaller than 0,045 cm.

In order to demonstrate this I have calculated for $\delta = 0{,}05$ two menisci, for which I have not accepted $\frac{H}{s} = 0{,}0354$ cm², which number may be derived from the data of QUINCKE for mercury, three hours after the formation of a drop [1]), but 0,0433 cm². for mercury, immediately after the formation of the drop. Then we have for:

$$R = 0{,}588 \text{ cm.} \quad f = 0{,}0168,$$

$$R = 0{,}455^5 \quad \text{»} \quad f = 0{,}0125;$$

while for $\frac{H}{s} = 0{,}0354$ cm². we get for:

$$R = 0{,}588 \text{ cm.} \quad f = 0{,}0172,$$

$$R = 0{,}455^5 \quad \text{»} \quad f = 0{,}0127.$$

And so we may easily complete the direct measurements by the limiting cases calculated on the supposition $\frac{H}{s} = 0{,}0354$ cm². up to the surface of interpolation. From § 7 it will appear that this value may certainly be put in stead of that which existed with the menisci observed by us.

I will now first draw the curve which represents f as a function of δ with the tube of 0,283 cm. radius (curve I in fig. VI of the plate) [2]).

For this I have drawn δ from the point A in a horizontal direction for which 0,0025 = 1 mm. is taken and f in a vertical direction for which 0,0005 = 1 mm.

In this manner from the menisci measured the points B, C, D and E have been obtained; but here it must be borne in mind that the curve is not determined by these points themselves, but by the condition that B and E and in the same way C and D must always be situated at equal distances on either side (comp. §§ 2 and 3).

Further are computed by means of the yet unsimplified formula:

$$f = \delta R - \sum \frac{\alpha_n}{n+1} R^{2n}$$

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[1]) But even this number is far from being certain, for from two kinds of series of experiments at 20° C., QUINCKE found also values corresponding to $\frac{H}{s} = 0{,}0391$ and $\frac{H}{s} = 0{,}0396$ cm².

[2]) Given in the Proceedings Dec. 1900.

the following values:

$\delta = 0{,}0991$; $f = 0{,}0148$ represented by the point F;

$\delta = 0{,}0708$; $f = 0{,}0105$ » » » » G;

$\delta = 0{,}0425$; $f = 0{,}00627$ » » » » H.

And then the line I is drawn.

In the same way line II in fig. VI is obtained for the tube of 0,382 cm. radius. From the point A, δ and f have been drawn in a similar manner and so we get the points L, M, N and O, for which the paired points are again L and N, together with M and O. The points P and Q have again been calculated.

Line III in the same fig. VI applies to the tube of 0,5814 cm. radius, and has been drawn from the point A. Here the paired points are S and T, and also U and V; W and X have been calculated. The points S' and T' as well as U' and V' belong to measurements in a tube of about the same width. It is difficult to draw the line through W and X and also between the paired points. But as I do not use tubes of more than 0,4 cm. radius, I have not considered this much further, because in such wide tubes the rim is no longer perfectly circular and parallax can not easily be avoided in the measurements.

Then fig. IV is drawn in which f as a function of R has always been drawn for the same value of δ. The scale values are again for f: $0{,}0005 = 1$ mm. and for R: $0{,}0025 = 1$ mm.

First we have drawn the points with $R = 0{,}2832$ cm. in the line I for the values of δ: 0,05; 0,1; 0,15; 0,2; 0,25; 0,3; 0,35; 0,4; the straight line on which these points are situated is in fig. IV also numbered by *1*.

Secondly the points with $R = 0{,}382$ cm. in the line II for the same values of δ; the straight line is also marked *2*.

Then the points for $R = 0{,}04$ cm. and $R = 0{,}1$ cm. have been calculated and lastly a number of points are calculated according to the formula (II), all for $\delta = 0{,}05$.

The points Y and Z are those calculated with the value $\frac{H}{s} = 0{,}0433$ cm^2.

Now the line for $\delta = 0{,}05$ could be drawn, by which the type for the lines $\delta =$ constant is known. Moreover we could draw each time the beginnings of those lines at small value of R, and so they could be continued through the points given by the lines *1* and *2*.

The rest of fig. VI has been derived from fig. IV by seeking each time for the same value of R in fig. IV the corresponding values of δ and f, and by drawing them anew as in the case of the curves I, II and III in fig. VI.

Curve V in fig. VI belongs to the tube of 0,409 cm. radius, of which only one meniscus was measured. The remaining lines in fig. VI belong to 0,05; 0,1; 0,15; 0,2; 0,25; 0,3; 0,35; and 0,4 cm. radius.

§ 7. The form of the meridian section of the meniscus can, if $\frac{H}{s}$ were exactly known, also be found graphically in the way shown by Lord KELVIN [1]). For if ϱ is the radius of curvature at the top of the meniscus, r_1 the radius of curvature at the point P of the normal section perpendicular to the meridian plane and r_2 the radius of curvature in the meridian plane, then we can write the equation:

$$\frac{1}{r_2} = \frac{s}{H} h + \frac{2}{\varrho} - \frac{1}{r_1},$$

so that, if we start from the top with a given radius of curvature we can always calculate r_2 if we have accepted some value for for $\frac{s}{H}$. For this I have again taken the value 28,25, hence $\frac{H}{s} = 0{,}0354$ cm^2. and then all the values must be expressed in cm. And so fig. VIII has been drawn on a 10 times magnified scale, in which $\varrho = 0{,}8$ cm. has been taken [2]). For r_1 and h we have each time taken the values which they have at the starting point of each element of the meridian curve so that the curvature is sure to be too small. In the same way fig. IX has been drawn in which h has been taken, as it is at the end of each element, so that the curvature is too large.

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1) To a request to Prof. PERRY about the drawings of the menisci made after this method, Prof. PERRY answered that they were not published in the paper in the Transactions of the Royal Society of Edinburgh and were afterwards lost.

2) This drawing, as well as fig. IX was originally constructed on a 30 times magnified scale and the curve was not divided into four as in the figure, but in twenty-four elements; in the reproduction on a $^1/_1$ scale only four lines of construction have been drawn.

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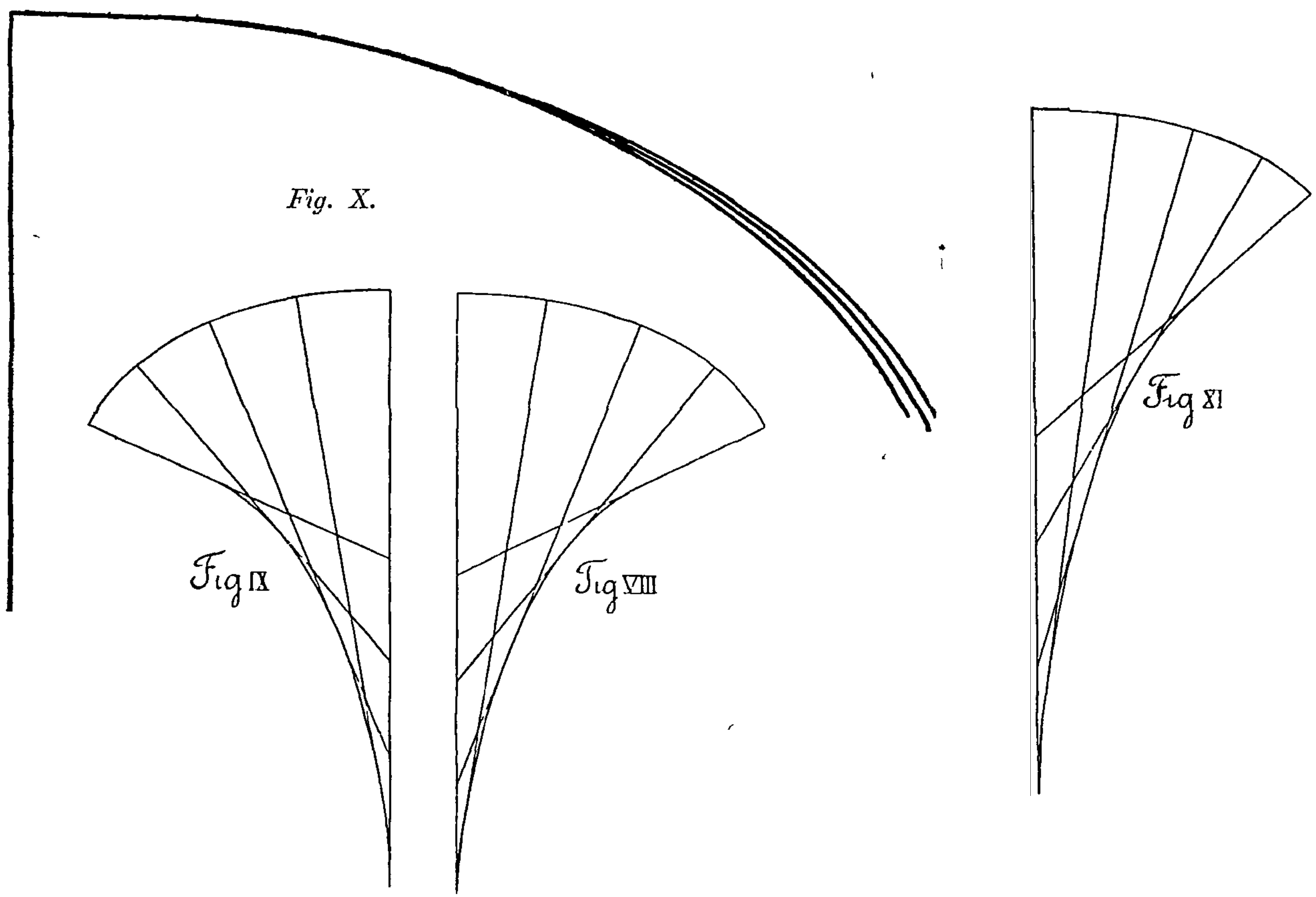

The two curves are combined in fig. X on the original 30 times magnified scale and there the mean curve has been drawn as a probable meridian section. Fig. XI represents the meniscus when the radius of curvature is 1,1 cm. at the top; it was drawn on a 25 times magnified scale, but is here again reproduced with some construction lines on a $^{25}/_{3}$ scale; while for h we have here always taken the height of the middle of each curve element and in the same way for r_1 the value, which that radius of curvature would have in the middle.

From the original drawings of the figures X and XI I have again calculated for several values of R (the radius of the tube), the height and the volume of the meniscus and from them again δ and f and I have also indicated these values in fig. VI by little squares; the deviation from the curves drawn already remains below the limit we require. The following values are found:

R.	p.	δ.	I.	f.	in fig. VI indicated by:
0,2	0,03	0,15	$\pi \times 0,000619$	0,0155	β
0,25	0,0487	0,195	$\pi \times 0,001582$	0,0253	γ
0,2832 (I)	0,0653	0,230	$\pi \times 0,00278$	0,0346	δ
»	0,0424	0,150	$\pi \times 0,00180$	0,0225	η
0,3	0,0758	0,253	$\pi \times 0,00367$	0,0408	ε
»	0,0493	0,164	$\pi \times 0,00238$	0,0264	θ
0,35	0,115	0,329	$\pi \times 0,00786$	0,0642	ζ
»	0,0727	0,208	$\pi \times 0,00484$	0,0395	ι
0,382 (II)	0,092	0,241	$\pi \times 0,00748$	0,0513	$\varkappa$
0,4	0,105	0,262	$\pi \times 0,00976$	0,0610	λ

§ 8. It follows from the given dimensions for menisci derived from the value $\frac{H}{s} = 0,0354$ cm². that the difference in value which $\frac{H}{s}$ has had in the menisci which I measured directly cannot have had much influence on the determination of the volume.

The mercury in the tubes used for that determination of the volume of the menisci was treated in exactly the same way as for the calibration of my piezometer tubes. And so we have as much certainty as can be obtained, that the values derived from the direct measurements of the menisci are applicable to the menisci which occur in the calibration.

Also for values of $\frac{H}{s}$ not deviating much from 0,0354 cm^2., as they may occur perhaps, when the piezometertubes are used with compressed gas, it will be allowable to use the values for the menisci which we have now found.

In general it is obvious that from the differential equation for h and r the same relation will be found when the unit of length is changed in the ratio of the square root of $\frac{H}{s}$. Thereby δ remains unchanged. If therefore $\frac{H}{s}$ changes from 0,0354 to 0,0433 cm^2., in order to be able to use the same values the unit of length must be taken $\sqrt{1,225}$ or 1,107 times larger. If for instance we desire to know f for $\delta = 0,35$ and $R = 0,3$ cm., $\frac{H}{s} = 0,0433$ cm^2. then we must look for it at $\delta = 0,35$ and $R = 0,271$ cm.; we then find $f = 0,0506$ and the value desired is 0,0560, while we find from the values measured: 0,0566; which would give a deviation of about 1 percent, and so within the limits we have indicated. For wider tubes the deviation increases; if for instance we want to know f for $\delta = 0,35$ and $R = 0,4$ cm., $\frac{H}{s} = 0,0433$ cm^2., then we find in Fig. IV at $\delta = 0,35$ and $R = 0,361$ cm., by continuing the curve a little $f = 0,0735$ and so the value sought is 0,0814; while from Fig. VI 0,0904 follows for the value measured, a large difference, for which it should be borne in mind that these numbers have not the accuracy of the values at a smaller δ, because they are obtained by continuing the curves for $R =$ const. and $\delta =$ const. a little beyond the range of observation. From the two instances given it appears that when H increases, for wide tubes ($R = 0,4$ cm.), the mean height decreases perceptibly. From the situation of the points ζ, $\varkappa$ and λ it would then follow that in the experiments $\frac{H}{s}$ would have been just a little smaller than 0,0354 cm^2. While as we see our results can be applied with a great certainty for the calibration, when we use compressed gas, this is dependent on the question how $\frac{H}{s}$ or as we must write that factor then: $\frac{H_{1,2}}{s_1-s_2}$ varies with the pressure of the gas. Corresponding to the important changes of $\frac{H}{s}$ arising from contact of the mercury surface with the air, the contact with a highly compressed gas can also influence it. As I could not obtain any indications on this point, I have assumed in my calculations that the influence of the pressure on $\frac{H_{1,2}}{s_1-s_2}$ may be neglected; it may be that later on we will be able to apply these corrections again.

That however these corrections will not probably become important for my determinations of isothermals, follows from the fact that the wide tube has only been used to 8 atm. for which the change of $\frac{H}{s}$ by the pressure will certainly be only very small; while at high pressures the volume is measured in narrower tubes, and we have proved that the influence of $\frac{H}{s}$ decreases as the tube becomes narrower.

§ 9. Although my research on the volume of the mercury meniscus has been made in order to evaluate the correction in the calibrations of our piezometertubes and in the measurements made by means of them, I have with a view to possible researches, for which the meniscus must be known still more accurately, read the *values of f* as accurately as possible in the figures IV and VI on the original drawing of which the scale was twice and a half as large again as that for the plate. We can now combine the values obtained in the following table; those which deviate imperceptibly from the mean height of the segment of a sphere have been printed in a small type.

To make it prominent for which menisci the deviation from a segment of a sphere begins to become important in our accurate determination of isothermals I have underlined them in the table [1]. The values obtained by extrapolation are in italics.

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[1]) In the calibration of the piezometertube of 0.4 cm. 15 menisci occurred, the heights of which varied from 0.087 ($\mathfrak{z} = 0.22$) cm. to 0.143 ($\mathfrak{z} = 0.36$) cm., mean height 0.114 ($\mathfrak{z} = 0.28^5$) cm.; in the measurements 80 menisci occurred from 0.092 ($\mathfrak{z} = 0,23$) cm. to 0.144 ($\mathfrak{z} = 0.36$) cm. height; most of them between 0.108 ($\mathfrak{z} = 0.27$) and 0.127

	δ							
R in c.m	0.05	0 1	0.15	0.2	0.25	0.3	0 35	0.4
0.05	0.0012^{6}	0.0025^{2}	0.0038	0.0050^{6}	0.00637	0.00773	0.0091	0.0107
0.1	0.0025^{3}	0.0050^{4}	0.0076	0.0102	0.0128	0.0155	0.0183	0.0213
0.15	0.0037^{6}	0.0075^{6}	0.0114^{5}	0.0153	0.0192^{5}	0.0232^{5}	0.0274	0.0318
0.2	0.0050^{5}	0.0103	0.0155	0.0206	0.0257	0.0310	0.0366	0.0426
0.25	0.0065^{5}	0.0131	0.0196^{5}	0.0261^{5}	0.0327	0.0393	0.0462	0.0536
0.3	0.0080	0.0159	0.0239	0.0320	0.0401	0.0483	0.0566	0.0657
0.35	0.0093^{5}	0.0188	0.0283	0.0384	0.0489	0.0592	0.0700	*0.0815*
0.4	0.0108^{5}	0.0218	0.0331	0.0453	0.0583	0.0737	*0.0904*	

I thought it better to let the table stand in this form, because on account of the slight curvature of the lines in fig. IV and VI a better interpolation is possible than if I had expressed the volume, in terms of the height and the radius.

But if many menisci at one width of the tube must be calculated, then tables must be derived for them from the preceding table.

If finally we reconsider the numerical example of § 1 we calculate from this table a section of 0,5 cm.^{2} and a height of 0,14 cm., a volume of 0,045 cc., while the segment of a sphere gives 0,0365 cc., and so we find a difference of about 0,0085 cc. or 23 percent, or more than 7 times the error allowed in our measurements, so that the correction calculated in these communications is indispensable for the accurate measurements aimed at.

($\delta = 0.32$) cm., on an average 0.115 ($\delta = 0.29$) cm. For the tube of 0.283 cm. radius I obtained in the calibration 16 menisci from $p = 0.042$ ($\delta = 0.15$) to $p = 0.095$ ($\delta = 0.33^{5}$) cm., on an average $p = 0.073$ ($\delta = 0.258$) cm.; in the measurements 33 menisci from $p = 0.031$ ($\delta = 0.11$) to $p = 0.121$ ($\delta = 0.43$) cm., on an average $p = 0.075$ ($\delta = 0.265$) cm. The third and the fourth tube are sufficiently narrow, so that we can omit the correction on the segment of sphere.

(February 20, 1901.)