As will be seen, ETARD has experimented here below 86° with unsaturated and above 86° with supersaturated solutions.

8. Finally, it is shown from the latest determinations which have been executed with great care and accuracy in the Physikalisch-Technische Reichsanstalt by DIETZ, FUNK, v. WROCHEM and MYLIUS [1]), that ETARD when he deteımined the solubility of the salts investigated by these authors has always worked with unsaturated solutions. In some cases the existing differences are not large. It is not worth while to insert here all the tables as the general result seems to be thoroughly established.

9. LENOBLE [2]) pointed out in 1896 that the data formerly communicated by ETARD did not lead to straight lines but to curves of the fourth degree or higher with slight curvature. As it now appears that ETARD's original material does not represent the true state of affairs, a closer investigation in this direction has become superfluous.

Result of the Investigation.

ETARD's law of solubility is not in agreement with the facts; a simple relation like this between the solubility of salts and the temperature does not seem to exist. Repetition of ETARD's experiments at high temperatures is desirable.

Amsterdam, Chem. Lab. University, February 1900.

Mathematics. — Prof. J. C. KLUYVER: "*On the expansion of a function in a series of polynomials.*"

According to BOREL's remark [3]) the fundamental problem consists in expanding $1:1-x$. For, having once obtained an expansion of the form

$$\frac{1}{1-x} = 1 + \sum_1^\infty T_n(x) = 1 + \sum_1^\infty \left(\alpha_{1n}\,x + \alpha_{2n}\,x^2 + \alpha_{3n}\,x^3 + \ldots \alpha_{nn}\,x^n\right)$$

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[1]) Wissenschaftliche Abhandlungen der Physikalisch-Technischen Reichsanstalt. Bd. III, 427.

[2]) Bulletin de la Société Chimique de Paris, XV (1896) 54.

[3]) Annales de l'école normale, t. 16, p. 132.

that can be rendered converging in every finite region of the x-plane, not enclosing any part of the straight line $(+1, +\infty)$, from

$$f(x) = \sum_0^\infty c_m (x-a)^m$$

we may deduce

$$f(x) = c_0 + \sum_1^\infty U_n(x-a) = c_0 + \sum_1^\infty \left[\alpha_{1n} c_1 (x-a) + \alpha_{2n} c_2 (x-a)^2 + \ldots \alpha_{nn} c_n (x-a)^n\right],$$

and the series of polynomials $U_n(x-a)$ can be made to represent $f(x)$ in every finite region of MITTAG-LEFFLER's "star".

Solutions of the fundamental problem are given by MITTAG-LEFFLER [1]), PAINLEVÉ [2]) and others; still as yet new solutions are not devoid of interest. Perhaps the solution described here is not behind in point of simplicity at least from a theoretical point of view.

As was shewn by PAINLEVÉ the problem of expanding $1:1-x$ is connected with a problem of conformal representation implying a certain want of determinateness. This problem requires the mapping of the interior of a u-circle, centre the origin and radius unity, on the interior of a nodeless closed z-curve, going round the origin and passing through $z=+1$. The homologues of $u=0$, $u=+1$ are to be the points $z=0$, $z=+1$; moreover the shape of the z-curve must be made to depend on a single or on several arbitrary parameters in such a manner, that by their assuming appropriate values the z-curve takes more or less elongated forms, varying from a z-circle, centre the origin, to an area of infinitesimal breadth covering the stroke $(0, +1)$.

In no other way is the choice of the z-curve limited. We take it here to be an ellipse having one focus in $z=0$ and the farther extremity of the axis major in $z=+1$.

SCHWARZ's functional relation

$$z = c \sin \left\{\frac{\pi}{2K} sn^{-1} \frac{u}{\sqrt{k}}\right\}$$

makes a u-circle and a z-ellipse conformal areas; since however by this formula the centres of both curves are corresponding points, and in our case the centre of the circle should be the homologue of one of the foci of the ellipse, a slight alteration is necessary.

[1]) Acta Mathematica, t. 23, p. 43 and t. 24, p. 183 and 205.
[2]) Comptes rendus, 23 May and 3 July 1899.

It will be seen that the correspondence defined by the equation

$$z = \frac{4q^{1/2}}{(1+q^{1/2})^2} \sin^2 \left\{ \frac{\pi}{2K} \, sn^{-1} \sqrt{\frac{u}{k}} \right\},$$

meets all requirements.

As to k, K and q, they are the usual JACOBIAN constants in the theory of elliptic functions; we will consider k and K as functions of q, thus making the latter quantity serve as an arbitrary real parameter able to assume all values between 0 and 1. Putting

$$\varepsilon = \frac{2q^{1/2}}{1+q},$$

the functional relation between u and z maps the u-circle on a z-ellipse represented in polar coordinates by the equation

$$R = \frac{1-\varepsilon}{1-\varepsilon \cos \varphi}.$$

When q tends to zero, the excentricity ε vanishes, the z-ellipse becomes a z-circle and ultimately we have $z = u$: on the contrary when q approaches its upper limit unity, the z-ellipse transforms itself into a narrow loop stretched round the stroke $(0, +1)$.

Obviously we may deduce from the functional relation an expansion of z in ascending powers of u. Writing

$$z = \frac{2q^{1/2}}{(1+q^{1/2})^2} \sum_1^\infty C_h \, u^h$$

the coefficients C_h are obtainable by means of the differential equation

$$u(k-u)(1-ku)\frac{d^2z}{du^2} + \frac{1}{2}\left[k - 2u(1+k^2) + 3ku^2\right]\frac{dz}{du} - \frac{\pi}{4K^2} z =$$

$$= \frac{2q^{1/2})}{(1+q^{1/2})^2} \cdot \frac{\pi}{4K^2}.$$

We shall find for the first and second terms [1])

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[1]) The notation of the ϑ-constants is that of TANNERY and MOLK, Éléments de la théorie des fonctions elliptiques.

$$C_1 = \frac{2}{\vartheta_2^2\,\vartheta_3^2}\,, \quad C_2 = \frac{2\,(\vartheta_2^4 + \vartheta_3^4 - 1)}{3\,\vartheta_2^4\,\vartheta_3^4}$$

and we may then use the relation

$$C_{h+1} = \frac{4\,[h^2\,(\vartheta_2^4 + \vartheta_3^4) - 1]}{(2\,h+1)\,(2\,h+2)\,\vartheta_2^2\,\vartheta_3^2}\,C_h - \frac{(h-1)\,(2\,h-1)}{(h+1)\,(2\,h+1)}\,C_{h-1}$$

to obtain the higher coefficients.

Similarly it is possible to expand z^n. For z^n as well as z itself is simply an aggregate of cosines of constant multiples of the quantity

$$\beta = \frac{\pi}{2K}\,sn^{-1}\sqrt{\frac{u}{k}}\,,$$

and the expansion of $\cos 2\,m\,\beta$ gives no more trouble than that of $\cos 2\,\beta$. In particular it should be noticed that the series for z^n begins with the term u^n.

The foregoing considerations enable us to express the function $1:1-xz$ as an integral series of u. For, in fact, we have only to expand the different powers of z in the series

$$1 + x\,z + x^2\,z^2 + x^3\,z^3 + \ldots.$$

and to arrange the result according to ascending powers of u.

In this way we obtain an expansion of the form

$$\frac{1}{1-xz} = 1 + \sum_1^{\infty} T_n\,(x, q)\,u^n,$$

where the coefficient $T_n\,(x,\,q)$ is a polynomial in x of order n, the coefficients of the polynomial involving the parameter q.

Putting now $x = \varrho e^{i\theta}$ we will ask under what conditions as to x and to q this u-series has its radius of convergence at least equal to unity. This point is examined in the following way. Suppose u to move at random through the interior of the u-circle, centre the origin and radius unity, then z simultaneously moves within the corresponding z-ellipse and the motion of the point xz is restricted to take place in the interior of a second ellipse of the z-plane. Evidently this xz-ellipse is obtained by turning the z-ellipse round $z=0$ through an angle θ, stretching at the same time its radii vectores in the ratio $\varrho:1$. Hence if only the point $z=1$ lies outside this xz-ellipse, given by the equation

$$R = \frac{(1-\varepsilon)\,\varrho}{1-\varepsilon \cos(\phi-\theta)},$$

the function $1:1-xz$ remains uniform and finite, whatever may be the position of u within the u-circle or even on its boundary. Therefore as soon as x and q be such that

$$1 > \frac{(1-\varepsilon)\,\varrho}{1-\varepsilon \cos\theta},$$

or what is the same that

$$\varrho < \frac{1}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\cos\theta,$$

the u-series will converge unconditionally for $|\,u\,| \leqq 1$.

We assume x and q to satisfy the condition imposed upon them and put $u = +1$; thus we obtain

$$\frac{1}{1-x} = 1 + \sum_{1}^{\infty} T_n(x, q),$$

a development of $1:1-x$ holding good for all points x inside the limaçon

$$\varrho = \frac{1}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\cos\theta.$$

This limaçon has its acnodal point in $x = 0$ and the nearer vertex in $x=1$. Its shape depends on the value of q; by variating this parameter we may regulate to a certain extent the region of convergence of the series of polynomials. Take $q = 0$ and the limaçon degenerates into a circle, centre $x = 0$, radius unity. Suppose q tending to its upper limit and the limaçon covers larger and larger parts of the x-plane. Ultimately for $q = 1$ the limaçon would enclose all points x of the plane except those lying on the straight line $(+1, +\infty)$.

Thus we infer that the expansion of $1:1-x$ can be made to converge in every finite region of the plane not including a part of the line $(+1, +\infty)$ and we may use it in the manner indicated at the beginning to form an expansion representing a given function $f(x)$.

So, for instance, taking $q = e^{-\pi}$, we have for all points x inside the limaçon

$$\varrho = 1.663 - 0.663 \cos\theta,$$

$$\frac{1}{1-x} = 1 + [0.5785\,x] + [0.2133\,x + 0.3347\,x^2] +$$
$$+ [0.0968\,x + 0.2468\,x^2 + 0.1936\,x^3] +$$
$$+ [0.0488\,x + 0.1575\,x^2 + 0.2142\,x^3 + 0.1120\,x^4] +$$
$$+ [0.0262\,x + 0.0978\,x^2 + 0.1762\,x^3 + 0.1652\,x^4 + 0.0648\,x^5] +$$

. . . .

If we now multiply the coefficients of $x^0, x^1, \ldots x^5, \ldots$ respectively by $0, 1, 0, -\frac{1}{3}, 0, \frac{1}{5}, \ldots$, that is by the corresponding coefficients of the power series

$$\lg\,\mathrm{tg}\,x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} \cdots,$$

we obtain the expansion

$$\lg\,\mathrm{tg}\,x = [0.5785\,x] + [0.2133\,x] + [0.0968\,x - 0.0645\,x^3] +$$
$$+ [0.0488\,x - 0.0714\,x^3] + [0.0262\,x - 0.0587\,x^3 + 0.0130\,x^5] + \ldots,$$

and the equivalence of the function and the series is valid for all points common to the interiors of the limaçons

$$\varrho = 1.663 \pm 0.663 \sin\theta.$$

And again in the same way we deduce from

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \frac{63}{256}x^5,$$

$$\frac{1}{\sqrt{1-x}} = 1 + [0.2892\,x] + [0.1066\,x + 0.1255\,x^2] +$$
$$+ [0.0484\,x + 0.0925\,x^2 + 0.0605\,x^3] +$$
$$+ [0.0244\,x + 0.0592\,x^2 + 0.0669\,x^3 + 0.0306\,x^4] +$$
$$+ [0.0131\,x + 0.0367\,x^2 + 0.0551\,x^3 + 0.0452\,x^4 + 0.0159\,x^5] + \ldots,$$

the region of convergence being the same as for the expansion of $1:1-x$.

For a test we may make the substition $x=-1$; we shall find

$$\frac{1}{2} = 0.5000 = 1-0.5785+0.1214-0.0436+0.0065-0.0042+\ldots$$
$$= 0.5016+\ldots,$$

$$bgtg(-1) = -0.7854 = -0.5785-0.2133-0.0323+$$
$$+0.0266+0.0195+\ldots = -0.7820+\ldots,$$

$$\frac{1}{\sqrt{2}} = 0.7070 = 1-0.2892+0.0189-0.0164-0.0015-$$
$$-0.0022+\ldots = 0.7096+\ldots.$$

Phycics. — Prof. J. D. VAN DER WAALS on: "*The equation of state and the theory of cyclic motion.*" II. (Continued from page 528).

Before we are able to calculate the equation for the equilibrium and the entropy and the specific heat of a substance with triatomic molecules, we must first know the mode of motion. If the motion should be such, that the first atom is placed exactly in the centre of gravity, and consequently only the two other atoms move, such a molecule must be regarded as a diatomic one, and the equation of the equilibrium will be again equal to:

$$\left(p+\frac{dP_v}{dv}+\frac{dP_b}{db}\right)(b-b_0) = RT.$$

But the value represented by b_0 will include besides the space of the moving atoms, also the space occupied by the stationary atom.

If the motion of the three atoms relative to their centre of gravity should be such that the distance of one of them quite determines the place of the two others, as would be the case when they move along three lines, which enclose constant angles, and if the case is therefore to be considered as a vibrating system with one degree of freedom, then such a molecule must be treated in our considerations as a diatomic one.