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The following papers were read:

Physics. — Prof. H. A. LORENTZ: "BOLTZMANN's *and* WIEN's *Laws of Radiation.*"

(Read February 23, 1901)

The theoretical proof of the laws, to which BOLTZMANN [1]) and WIEN [2]) have been led by the application of thermodynamics to the phenomena of radiation may be made to depend directly on the equations of the electromagnetic field, a method which has the

[1]) BOLTZMANN, Wied. Ann. Bd. 22, p. 291; 1884.
[2]) WIEN, Berliner Sitz. Berichte, 1893, p. 55.

advantage that the notion of "rays" of light and heat is almost wholly avoided.

§ 1. Let us consider a space, enclosed by walls that are perfectly reflecting on the inside, and containing a ponderable body M, the remaining part being occupied by aether. In this medium we shall then have a state of radiation, the nature of which is determined by the temperature T of the body M; in virtue of this state the aether will exert on the reflecting walls a certain pressure, the amount of which for unit area we shall denote by p. Let v be the volume within the enclosure. It may be enlarged or diminished by a displacement of the walls. We shall also suppose that by some means or other heat may be imparted to the body M.

Now, choosing v and T as independent variables, and denoting by ε the energy of the whole system, we shall have

$$dQ = \frac{\partial \varepsilon}{\partial T} dT + \left(\frac{\partial \varepsilon}{\partial v} + p\right) dv$$

for the heat that is required for the infinitesimal change, determined by dT and dv, and, by the rule that $\frac{dQ}{T}$ is an exact differential,

$$\frac{\partial \varepsilon}{\partial v} + p = T \frac{\partial p}{\partial T}.$$

Here the first term represents the energy of the aether per unit volume, which we shall call U. Indeed, if we increase the volume v, keeping the temperature constant, the ponderable body will remain in the same state (the pressure p exerted on this body by the surrounding aether will not be altered, being a function of T alone); the increment of ε will therefore be the energy contained in the new part that is added to v. Hence

$$U + p = T \frac{dp}{dT}, \quad \ldots \ldots \ldots \quad (1)$$

the last term containing an ordinary differential coefficient, because p is independent of v.

§ 2. We shall combine this result with the simple relation

$$p = \frac{1}{3} U \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which we now proceed to prove. To this effect we remember in the first place that the energy per unit volume is given by [1])

$$2 \pi V^2 \mathfrak{d}^2 + \frac{1}{8\pi} \mathfrak{H}^2.$$

We shall therefore write

$$U = 2 \pi V^2 \overline{\mathfrak{d}^2} + \frac{1}{8\pi} \overline{\mathfrak{H}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the horizontal bars indicating mean values with respect to place and time, which we might calculate by computing in the first place the mean values for all points of a certain space, and by taking then, for a certain lapse of time, the mean of these space-means. In this it is to be understood that the dimensions of the space in question and the length of the lapse of time have to be large, as compared with the wave-length and the time of vibration.

If we confine ourselves to such mean values, the forces acting on the walls may be regarded as due to a state of stress in the aether. If α, β and γ are the direction-cosines of the normal n of an element of surface, the first component of the stress on this element will be

$$X_n = 2 \pi V^2 (2 \overline{\mathfrak{d}_x \mathfrak{d}_n} - \alpha \overline{\mathfrak{d}^2}) + \frac{1}{8\pi} (2 \overline{\mathfrak{H}_x \mathfrak{H}_n} - \alpha \overline{\mathfrak{H}^2});$$

i.e., this will be the force in the direction of OX, exerted by the part of the medium which lies on the side of the element, indicated by the normal n.

Now, the state of radiation we are considering has the same properties in all directions. From this it follows that there are no tangential stresses and that the normal stress is the same for all directions of the element of surface. It is given by

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[1]) The notation is the same as in my »Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern", from which memoir I have also borrowed several formulae.

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$$X_x = 2\pi V^2 (2\overline{\mathfrak{d}_x^2} - \overline{\mathfrak{d}^2}) + \frac{1}{8\pi}(2\overline{\mathfrak{H}_x^2} - \overline{\mathfrak{H}^2}).$$

But, in an *isotropic* state,

$$\overline{\mathfrak{d}_x^2} = \overline{\mathfrak{d}_y^2} = \overline{\mathfrak{d}_z^2}, \qquad \overline{\mathfrak{H}_x^2} = \overline{\mathfrak{H}_y^2} = \overline{\mathfrak{H}_z^2}.$$

Therefore:

$$\overline{\mathfrak{d}_x^2} = \frac{1}{3}\overline{\mathfrak{d}^2}, \qquad \overline{\mathfrak{H}_x^2} = \frac{1}{3}\overline{\mathfrak{H}^2},$$

and

$$X_x = -\frac{2}{3}\pi V^2 \overline{\mathfrak{d}^2} - \frac{1}{24\pi}\overline{\mathfrak{H}^2}.$$

In comparing this formula, in which the negative sign indicates a *pressure*, with (3), we arrive at the relation (2).

In virtue of this the equation (1) now takes the form:

$$4U = T\frac{dU}{dT},$$

and so we find the law, enunciated by BOLTZMANN, that the energy U per unit volume is proportional to the fourth power of the absolute temperature.

§ 3. If the volume v is increased, the system will do an external work and a larger volume of aether will be filled with the energy of radiation; for both reasons the temperature of the body M will sink, if the operation is conducted adiabatically. We may also, before increasing the volume, remove the body M; in this case we start from a volume v of aether in the particular state of radiation that corresponds to the temperature T, and we get new states by letting the walls recede with a velocity which we shall suppose to be extremely small in comparison with the velocity of light. Now WIEN has shown in the first place, by a train of thermodynamical reasoning, that these new states, of smaller energy-density than the original one, are precisely such as can be in equilibrium with ponderable bodies of temperatures lower than T. Using BOLTZMANN's law, we may express this as follows: After having diminished, by means of an adiabatic expansion, the energy per unit volume from U to U', we shall have arrived at a state of radiation which may be in equilibrium with a ponderable body of the temperature

$$T' = T\sqrt[4]{\frac{U'}{U}}.$$

This theorem, which I shall here admit without further discussion, enables us to determine the relation between the states of radiation corresponding to the temperatures T and T'. For this purpose it will only be necessary to compare the states of the aether before and after the expansion. This is the second part of the proof given by WIEN, and it is this part we shall present in a modified form by applying the well known equations of the electromagnetic field to the phenomena in the aether within the receding walls. If we suppose the expanding enclosure to remain geometrically similar to itself, the problem may be treated by the introduction of a suitable set of new variables. In seeking for these, I have kept in mind the substitutions that had proved of use in the theory of aberration, a theory in which we have likewise to do with moving ponderable bodies. Of course there is a difference between the two cases; in the problem of aberration the velocity is the same for all bodies concerned, whereas, in the question now under consideration, it is unequal for different points of the enclosures.

§ 4. I shall suppose the dilatation of the walls to be equal in all directions, and to have the same amount in equal infinitely small times. This may be expressed by assuming

$$x = x' e^{at},\ y = y' e^{at},\ z = z' e^{at}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

with a constant value of a, as the relation between the coordinates x, y, z of a point of the walls at time t, and the coordinates x', y', z' of the same point at the instant $t = 0$, at which we begin to consider the phenomena. Indeed, the velocities are by (4)

$$ax,\quad ay,\quad az; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

during the time dt the linear dimensions will therefore be changed in the ratio of 1 to $1 + adt$.

As to the constant a, we shall take it so small that the velocities (5) are extremely small in comparison with the velocity of light. Notwithstanding this we may, by sufficiently increasing t, assign to the factor e^{at} any value we like.

After having assumed for the walls the formulae (4), it is natural to replace the coordinates x, y, z of any point of the enclosed space by the new variables

$$x' = x\,e^{-at},\ y' = y\,e^{-at},\ z' = z\,e^{-at}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The fourth independent variable, the time t, will likewise be replaced by a new one. For this we take[1])

$$t' = \frac{1}{a}(1-e^{-at}) - \frac{a}{2\,V^2}(x^2+y^2+z^2)\,e^{-at}. \quad . \quad . \quad . \quad (7)$$

The dependent variables which occur in the equations of the electromagnetic field are now to be considered as functions of x', y', z', t'. In doing so, we have to use the relations

$$\left.\begin{aligned}
\frac{\partial}{\partial x} &= e^{-at}\frac{\partial}{\partial x'} - \frac{ax}{V^2}e^{-at}\frac{\partial}{\partial t'},\\
\frac{\partial}{\partial y} &= e^{-at}\frac{\partial}{\partial y'} - \frac{ay}{V^2}e^{-at}\frac{\partial}{\partial t'},\\
\frac{\partial}{\partial z} &= e^{-at}\frac{\partial}{\partial z'} - \frac{az}{V^2}e^{-at}\frac{\partial}{\partial t'},
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{\partial}{\partial t} = e^{-at}\left[1+\frac{a^2}{2\,V^2}(x^2+y^2+z^2)\right]\frac{\partial}{\partial t'} -$$

$$- ax\,e^{-at}\frac{\partial}{\partial x'} - ay\,e^{-at}\frac{\partial}{\partial y'} - az\,e^{-at}\frac{\partial}{\partial z'}. \quad . \quad . \quad . \quad (9)$$

The variables which serve to determine the state of the aether are $\mathfrak{d}_x$, $\mathfrak{d}_y$, $\mathfrak{d}_z$, $\mathfrak{H}_x$, $\mathfrak{H}_y$, $\mathfrak{H}_z$. We shall replace these by the quantities $\mathfrak{d}'_x$, $\mathfrak{d}'_y$, $\mathfrak{d}'_z$, $\mathfrak{H}'_x$, $\mathfrak{H}'_y$, $\mathfrak{H}'_z$, which are defined by the following equations[2])

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[1]) As regards the last term, this value of t' is an imitation of the expression for the "local time", which I have introduced into the theory of aberration (l. c. p. 49).

[2]) The latter terms in these equations correspond to similar terms in the equations of the theory of aberration.

$$\left.\begin{aligned}\mathfrak{d}_x &= e^{-2at}\,\mathfrak{d}'_x - \frac{a}{4\pi V^2}(y\,\mathfrak{H}_z - z\,\mathfrak{H}_y),\\ \mathfrak{d}_y &= e^{-2at}\,\mathfrak{d}'_y - \frac{a}{4\pi V^2}(z\,\mathfrak{H}_x - x\,\mathfrak{H}_z),\\ \mathfrak{d}_z &= e^{-2at}\,\mathfrak{d}'_z - \frac{a}{4\pi V^2}(x\,\mathfrak{H}_y - y\,\mathfrak{H}_x),\end{aligned}\right\} \quad . \; . \; . \; (10)$$

$$\left.\begin{aligned}\mathfrak{H}_x &= e^{-2at}\,\mathfrak{H}'_x + 4\pi a\,(y\,\mathfrak{d}_z - z\,\mathfrak{d}_y),\\ \mathfrak{H}_y &= e^{-2at}\,\mathfrak{H}'_y + 4\pi a\,(z\,\mathfrak{d}_x - x\,\mathfrak{d}_z),\\ \mathfrak{H}_z &= e^{-2at}\,\mathfrak{H}'_z + 4\pi a\,(x\,\mathfrak{d}_y - y\,\mathfrak{d}_x).\end{aligned}\right\} \quad . \; . \; . \; . \; (11)$$

It must be kept in mind that the coefficient a is very small. Let l be the largest value of any dimension of the system during the interval of time we wish to consider. Then, by our assumption,

$$\frac{a\,l}{V}$$

has a very small value $\varkappa$; evidently, $\frac{ax}{V}, \frac{ay}{V}, \frac{az}{V}$ will be of the same order of magnitude. Hence, if we neglect quantities which, compared with other terms in the same equation, are of the order $\varkappa^2$, we may omit in (9) the term containing a^2. By this, the relation becomes

$$\frac{\partial}{\partial t} = e^{-at}\frac{\partial}{\partial t'} - ax\,e^{-at}\frac{\partial}{\partial x'} - ay\,e^{-at}\frac{\partial}{\partial y'} - az\,e^{-at}\frac{\partial}{\partial z'}. \quad . \; . \; (9')$$

We may add that for vibratory disturbances of the natural state of the aether, the operations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are comparable to $\frac{1}{V}\frac{\partial}{\partial t}$, as regards the order of magnitude of the result, and that $\mathfrak{H}$ is of the same order as $V\mathfrak{d}$. From this it follows that, in the equations (8), (9'), (10) and (11), all terms containing the factor a are of the order $\varkappa$, relatively to the terms in which a does not occur. Similarly, attentive consideration of the formulae that will be deduced in the next article shows that, in comparison with the terms without a, all those which contain the factor a^2 are of the order $\varkappa^2$. We shall therefore neglect all terms in which a^2 appears.

§ 5. The first equation of motion is:

$$\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} = 4\pi \frac{\partial \mathfrak{d}_x}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Putting for $\mathfrak{H}_y$ and $\mathfrak{H}_z$ the values (11), we find for its left-hand side:

$$e^{-2at}\left(\frac{\partial \mathfrak{H}'_z}{\partial y} - \frac{\partial \mathfrak{H}'_y}{\partial z}\right) + 4\pi a\left(x\frac{\partial \mathfrak{d}_y}{\partial y} - y\frac{\partial \mathfrak{d}_x}{\partial y} - \mathfrak{d}_x\right) -$$

$$- 4\pi a\left(z\frac{\partial \mathfrak{d}_x}{\partial z} - x\frac{\partial \mathfrak{d}_z}{\partial z} + \mathfrak{d}_x\right).$$

Since

$$\frac{\partial \mathfrak{d}_x}{\partial x} + \frac{\partial \mathfrak{d}_y}{\partial y} + \frac{\partial \mathfrak{d}_z}{\partial z} = 0,$$

we may also write for it

$$e^{-2at}\left(\frac{\partial \mathfrak{H}'_z}{\partial y} - \frac{\partial \mathfrak{H}'_y}{\partial z}\right) - 8\pi a\,\mathfrak{d}_x - 4\pi a\left(x\frac{\partial \mathfrak{d}_x}{\partial x} + y\frac{\partial \mathfrak{d}_x}{\partial y} + z\frac{\partial \mathfrak{d}_x}{\partial z}\right),$$

and, if we neglect terms with a^2,

$$e^{-2at}\left(\frac{\partial \mathfrak{H}'_z}{\partial y} - \frac{\partial \mathfrak{H}'_y}{\partial z}\right) - 8\pi a\,e^{-2at}\,\mathfrak{d}'_x -$$

$$- 4\pi a\,e^{-3at}\left(x\frac{\partial \mathfrak{d}'_x}{\partial x'} + y\frac{\partial \mathfrak{d}'_x}{\partial y'} + z\frac{\partial \mathfrak{d}'_x}{\partial z'}\right). \quad . \quad . \quad . \quad (13)$$

Again, using the same simplification,

$$\frac{\partial}{\partial y} = e^{-at}\,\frac{\partial}{\partial y'} - \frac{ay}{V^2}\frac{\partial}{\partial t},$$

$$\frac{\partial}{\partial z} = e^{-at}\,\frac{\partial}{\partial z'} - \frac{az}{V^2}\frac{\partial}{\partial t},$$

$$\frac{\partial \mathfrak{H}'_z}{\partial y} - \frac{\partial \mathfrak{H}'_y}{\partial z} = e^{-at}\left(\frac{\partial \mathfrak{H}'_z}{\partial y'} - \frac{\partial \mathfrak{H}'_y}{\partial z'}\right) - \frac{a}{V^2}\left(y\frac{\partial \mathfrak{H}'_z}{\partial t} - z\frac{\partial \mathfrak{H}'_y}{\partial t}\right).$$

In the last term of this equation, $\frac{\partial \mathfrak{H}'_y}{\partial t}$ and $\frac{\partial \mathfrak{H}'_z}{\partial t}$ may be replaced by $e^{2at}\frac{\partial \mathfrak{H}_y}{\partial t}$ and $e^{2at}\frac{\partial \mathfrak{H}_z}{\partial t}$, as appears from (11). The expression (13) therefore becomes

$$e^{-3at}\left(\frac{\partial \mathfrak{H}'_z}{\partial y'}-\frac{\partial \mathfrak{H}'_y}{\partial z'}\right)-\frac{a}{V^2}\left(y\frac{\partial \mathfrak{H}_z}{\partial t}-z\frac{\partial \mathfrak{H}_y}{\partial t}\right)-$$

$$-8\pi a e^{-2at}\mathfrak{d}'_x-4\pi a e^{-3at}\left(x\frac{\partial \mathfrak{d}'_x}{\partial x'}+y\frac{\partial \mathfrak{d}'_x}{\partial y'}+z\frac{\partial \mathfrak{d}'_x}{\partial z'}\right). \quad (14)$$

The right-hand side of (12) is, by (10),

$$-8\pi a e^{-2at}\mathfrak{d}'_x+4\pi e^{-2at}\frac{\partial \mathfrak{d}'_x}{\partial t}-\frac{a}{V^2}\left(y\frac{\partial \mathfrak{H}_z}{\partial t}-z\frac{\partial \mathfrak{H}_y}{\partial t}\right),$$

or, if (9') is taken into account,

$$-8\pi a e^{-2at}\mathfrak{d}'_x+4\pi e^{-3at}\frac{\partial \mathfrak{d}'_x}{\partial t'}-$$

$$-4\pi a e^{-3at}\left(x\frac{\partial \mathfrak{d}'_x}{\partial x'}+y\frac{\partial \mathfrak{d}'_x}{\partial y'}+z\frac{\partial \mathfrak{d}'_x}{\partial z'}\right)-$$

$$-\frac{a}{V^2}\left(y\frac{\partial \mathfrak{H}_z}{\partial t}-z\frac{\partial \mathfrak{H}_y}{\partial t}\right). \quad . \quad . \quad . \quad (15)$$

Finally we shall find, instead of (12), after division by e^{-3at},

$$\frac{\partial \mathfrak{H}'_z}{\partial y'}-\frac{\partial \mathfrak{H}'_y}{\partial z'}=4\pi\frac{\partial \mathfrak{d}'_x}{\partial t'}.$$

The other equations may be treated in the same way and all relations between the new variables will be found to be of the same form as those between the original ones.

§ 6. We have also to attend to the surface conditions at the walls. These latter will be perfectly reflecting, if made of a substance of infinite specific inductive capacity, and then, if the wall is at rest, the tangential components of the dielectric displacement in the adjacent aether will be zero. Therefore, if

$$F(x, y, z) = 0 \quad \ldots \ldots \ldots \quad (16)$$

is the equation of the wall, we shall have

$$\mathfrak{d}_x : \mathfrak{d}_y : \mathfrak{d}_z = \frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z} \quad \ldots \ldots \quad (17)$$

In examining the phenomena of aberration, I have had occasion to consider the conditions that have to be fulfilled at the surface of separation of two bodies. These latter were supposed to move with a common velocity $\mathfrak{p}$, and it was found that all equations, the surface conditions as well as those for the interior of the bodies, might, by an appropriate choice of new variables, be reduced to the form that holds in the case of bodies at rest. Instead of the dielectric displacement with the components

$$\mathfrak{d}_x,\ \mathfrak{d}_y,\ \mathfrak{d}_z\ , \quad \ldots \ldots \ldots \quad (18)$$

a new vector with the components

$$\left.\begin{aligned} &\mathfrak{d}_x + \frac{1}{4\pi V^2}(\mathfrak{p}_y\,\mathfrak{H}_z - \mathfrak{p}_z\,\mathfrak{H}_y), \\ &\mathfrak{d}_y + \frac{1}{4\pi V^2}(\mathfrak{p}_z\,\mathfrak{H}_x - \mathfrak{p}_x\,\mathfrak{H}_z), \\ &\mathfrak{d}_z + \frac{1}{4\pi V^2}(\mathfrak{p}_x\,\mathfrak{H}_y - \mathfrak{p}_y\,\mathfrak{H}_x) \end{aligned}\right\} \quad \ldots \ldots \quad (19)$$

was introduced. Hence, it will be this new vector, whose tangential components must vanish at a moving perfectly reflecting surface.

Let us apply this rule to an element of the walls of the expanding enclosure. The velocity-components $\mathfrak{p}_x$, $\mathfrak{p}_y$, $\mathfrak{p}_z$ must now be replaced by $a\,x$, $a\,y$, $a\,z$. Using at the same time the formulae (10), we find for the expressions (19)

$$e^{-2at}\,\mathfrak{d}'_x,\ e^{-2at}\,\mathfrak{d}'_y,\ e^{-2at}\,\mathfrak{d}'_z .$$

It thus appears that the vector

$$e^{-2at}\,\mathfrak{d}'$$

must be perpendicular to the wall. The vector $\mathfrak{d}'$ must be so likewise, so that, if at any moment

$$F'(x, y, z) = 0$$

is the equation of the walls, we shall have

$$\mathfrak{d}'_x : \mathfrak{d}'_y : \mathfrak{d}'_z = \frac{\partial F'}{\partial x} : \frac{\partial F'}{\partial y} : \frac{\partial F'}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now, if at the instant $t = 0$, the walls coincide with the surface determined by (16), the equation at any later time will be

$$F(x', y', z') = 0,$$

where

$$x' = x\, e^{-at},\ y' = y\, e^{-at},\ z' = z\, e^{-at},$$

agreeing with (4). Thus:

$$F'(x, y, z) = F(x', y', z'),$$

and, if we differentiate for a constant t,

$$\frac{\partial F'}{\partial x} : \frac{\partial F'}{\partial y} : \frac{\partial F'}{\partial z} = \frac{\partial F}{\partial x'} : \frac{\partial F}{\partial y'} : \frac{\partial F}{\partial z'},$$

so that the surface conditions become

$$\mathfrak{d}'_x : \mathfrak{d}'_y : \mathfrak{d}'_z = \frac{\partial F}{\partial x'} : \frac{\partial F}{\partial y'} : \frac{\partial F}{\partial z'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

On the right-hand side of this formula, x', y', z' occur in exactly the same manner as x, y, z in the formula (17).

§ 7. If the enclosure were permanently in the position it occupies at the time $t = 0$, $\mathfrak{d}_x$, $\mathfrak{d}_y$, $\mathfrak{d}_z$, $\mathfrak{H}_x$, $\mathfrak{H}_y$, $\mathfrak{H}_z$ would be certain functions of x, y, z, t, say

$$\mathfrak{d}_x = \varphi_1(x, y, z, t),\ \mathfrak{H}_x = \chi_1(x, y, z, t), \text{ etc.}; \quad . \quad . \quad (22)$$

these will satisfy both the equations of the field and the surface conditions (17).

Now, from all that has been said, it appears that the values

$$\mathfrak{d}'_x = \varphi_1 (x', y', z', t'),\ \mathfrak{H}_x' = \chi_1 (x', y', z', t'),\ \text{etc.} \ldots \quad (23)$$

will be a solution of the equations of the field, taken conjointly with the conditions (21), we have found for the receding walls. We have thus got expressions representing the state of the aether during the expansion.

Now, we shall especially consider the state of things, existing at the moment when the dimensions have become

$$e^{at} = k$$

times what they were originally. A definite value of this coefficient k may be reached in a shorter or a longer time, this depending on the value of a. We shall however consider the limit to which the state of the aether tends, if, while we keep k fixed, t is continually increased and a continually diminished. By (10) and (11) we shall have ultimately

$$\mathfrak{d} = \frac{\mathfrak{d}'}{k^2}, \text{ and } \mathfrak{H} = \frac{\mathfrak{H}'}{k^2};$$

therefore, at the limit,

$$\mathfrak{d}_x = \frac{1}{k^2}\varphi_1\left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}, t'\right),\ \mathfrak{H}_x = \frac{1}{k^2}\chi_1\left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}, t'\right),\ \text{etc.} \quad (24)$$

As to the variable t', it is related to t in a somewhat complicated manner; the relation between the differentials takes however the simple form

$$dt' = \frac{1}{k}dt.$$

It is easily seen that the function (24) will satisfy the surface conditions such as they are for walls that are kept at rest. This is what we might have expected. By sufficiently diminishing the velocity of the walls, we make the system pass through a series of successive states that may, each of them, be regarded as a state of equilibrium. By WIEN's principle (§ 3) we know already that each of these states might continue to exist if the enclosure contained a ponderable body of a definite temperature.

The series starts with the state (22), with which (24) coincides if $k = 1$; it then passes to increasing values of k.

We shall denote by T the temperature of a ponderable body that may be in equilibrium with (22), and by T' the corresponding temperature for (24).

§ 8. Let us now compare the states (22) and (24). At first sight there is a difficulty in as much as the variables t and t' have widely different values. It is to be borne in mind, however, that the state (22) is a *stationary* one; i. e. all particulars that may be deduced from observation are independent of the time t.

We may therefore begin by choosing the instant for which we wish to consider the state (24); a definite value having in this way been assigned to t', we may give an equal value to the time t in (22). In other words, we shall compare the quantities (24) with the values

$$\mathfrak{d}_x = \varphi_1 (x, y, z, t'),\ \mathfrak{H}_x = \chi_1 (x, y, z, t'),\ \text{etc.}, \quad . \quad . \quad . \quad (25)$$

the latter state existing in a certain space S, and the former in a space S', whose dimensions are k times as great.

The values of $\mathfrak{d}$ and $\mathfrak{H}$ in corresponding points of S and S' are to each other as 1 to $\frac{1}{k^2}$, and the energy per unit volume will be in (24) k^4 times smaller than in (25). Hence, remembering BOLTZMANN's law,

$$T' = \frac{T}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In examining the phenomena, represented by (25), it may be convenient to decompose, by means of FOURIER's theorem, or otherwise, the values (25) into functions of x, y, z of a less complicated form. After having accomplished such a decomposition for (25), a similar development of (24) may at once be written down. For instance, if

$$\psi_1 (x, y, z, t')$$

is one of the parts of $\mathfrak{d}_x$ in (25), the corresponding part in (24) will be

$$\frac{1}{k^2} \psi_1 \left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}, t'\right).$$

There is also a simple relation between the space-variations in the two cases. Let $P\,Q$ and $P'\,Q'$ be corresponding lines in S and S'. Then, if we denote by η one of the components of $\mathfrak{d}$ or $\mathfrak{H}$, and by η_P, η_Q, $\eta_{P'}$, $\eta_{Q'}$ its values in the points considered, we shall have

$$\frac{\eta_Q - \eta_P}{\eta_P} = \frac{\eta_{Q'} - \eta_{P'}}{\eta_{P'}} ;$$

i. e. the relative variations along corresponding lines will be equal.

From this it is immediately seen that, if one of the parts into which we have decomposed (25) is characterized by a definite wave-length l, the corresponding part of (24) will have a wave-length

$$l' = kl.$$

Therefore

$$l : l' = T' : T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

i. e. corresponding wave-lengths in the two states are to each other in the inverse ratio of the temperatures.

We have already spoken of the ratio between the values of the energy per unit volume. We may add that this ratio, equal to that of the fourth powers of the temperatures, does not only hold for the really existing states of motion, but also for the parts into which these may be decomposed in the way that has been indicated. If, in the state corresponding to the temperature T, there is a certain amount of energy u per unit volume, depending on the vibrations whose wave-lengths lie between certain limits, and if, in the state for which the temperature is T', u' is the energy per unit volume due to the vibrations of corresponding wave-lengths, we shall have

$$u : u' = T^4 : T'^4.$$

This equation, taken together with (27), is the expression of the law of WIEN.