Temp.	Tot Vol. 0,0	Pressure	Temp	Tot. Vol. 0,0	Pressure.
30°,23	1182	52,51	41°,45	1179	58,10
(contin.)	0993	55,61	(contin.)	0993	62,47
	0797	58,24		0800	67,27
	0560	60,48		0571	73,93
	0399	68,04	52°,5	2613	37,40
41°,45	2445	37,29		2428	39,59
	2285	39,12		2168	43,03
	2100	41,48		1921	46,90
	1910	44,18		1662	51,70
	1727	47,04		1187	63,13
	1545	50,42		0936	71,12
	1362	54,08		0758	77,77

Physics. — "*The elementary theory of the* ZEEMAN-*effect. Reply to an objection of* POINCARÉ." By Prof. H. A. LORENTZ.

§ 1. In a recent article in L'Éclairage Électrique [1]) POINCARÉ comes to the conclusion that the well known theory of ZEEMAN's phenomenon, according to which every luminous particle contains either a single movable ion, or a certain number of such ions whose vibrations are mutually independent, can account for the doublet which is seen along the lines of force, but is unable to explain the triplet which one observes in a direction perpendicular to these lines. This result is obtained by treating, not the emission but the absorption in the magnetic field, and it is curious that the same mode of reasoning has led VOIGT [2]) to formulae implying the existence of the triplet. I believe this discrepancy to be due to POINCARÉ's erroneously omitting the term

1) POINCARÉ, La théorie de LORENTZ et le phénomène de ZEEMAN, Éclairage Électrique, T. 19, p. 5, 1899.

2) VOIGT, Ueber den Zusammenhang zwischen dem ZEEMAN'schen und dem FARADAY'schen Phanomen, Gottinger Nachrichten, 1898, Heft 4, p. 1.

$$\varepsilon_k \, \alpha \, \frac{d\,Z_k}{d\,t}$$

in his equation (6) on page 8.

In order to explain this, I shall in the first place compare the different formulae that may be applied to the propagation of light in an absorbing gaseous body, placed in a magnetic field.

§ 2. The equations of VOIGT contain the following quantities:

1°. The components u, v, w of a vector (the vector of NEUMANN) which comes into play in all media, the aether itself included, which are traversed by light-waves.

2°. The components ξ, η, ζ of a second vector (the vector of FRESNEL), which is related to the former in the way expressed by the equations

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad . \quad . \quad . \quad (1)$$

3°. A certain number of vectors P_1, P_2, P_3, ..., serving to determine the state of the ponderable molecules, and each of them corresponding to one of the principal modes of vibration of a molecule. The components of the vector P_h are denoted by U_h, V_h, W_h, and the index h is likewise affixed to the constant coefficients belonging to these different vectors.

4°. A vector (Ξ, H, Z), which is defined as follows:

$$\Xi = \mathrm{v}^2 \xi + \Sigma\, \varepsilon_h\, U_h, \quad H = \mathrm{v}^2 \eta + \Sigma\, \varepsilon_h\, V_h, \quad Z = \mathrm{v}^2 \zeta + \Sigma\, \varepsilon_h\, W_h \, . \quad (2)$$

Here, the coefficients ε are constants, and v is the velocity of light in the aether [1]).

Between the vectors (Ξ, H, Z) and (u, v, w) there exists a relation, expressed by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial H}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{\partial^2 v}{\partial t^2} = \frac{\partial Z}{\partial x} - \frac{\partial \Xi}{\partial z}, \quad \frac{\partial^2 w}{dt^2} = \frac{\partial \Xi}{\partial y} - \frac{\partial H}{\partial x} \quad . \quad . \quad (3)$$

Finally there are a certain number of equations — three for each

1) In order to avoid confusion, I shall depart a little from the notation of VOIGT and from that which I myself have used on former occasions.

vector P_h — which are to be considered as the equations of motion for the ponderable matter. They are of the form

$$\frac{\partial^2 U_h}{\partial t^2} + d_h U_h + f_h \frac{\partial U_h}{\partial t} + g_h \left(B \frac{\partial W_h}{\partial t} - C \frac{\partial V_h}{\partial t}\right) + \varepsilon_h \xi = 0, \text{ etc.}^{1)} \quad (4)$$

in which d, f and g are constants. The terms with the first coefficient depend on the elastic forces acting in the ponderable particles, the terms with f serve to introduce a resistance and consequently an absorption of light, while the terms with g are due to the forces produced by the magnetic field.

The field is supposed to be homogeneous; the components of the magnetic force in the field are A, B, C.

In the simplest case there is only one vector P. The signs of summation (in (2)) and the indices h are then to be omitted, and there will be no more than three equations (4).

§ 3. On the basis of the electromagnetic theory of light I have established the equations of motion in the following way [2]).

Let there be N equal molecules per unit of volume, each of them containing a movable ion of charge e and effective mass $\varkappa$. Let x, y, z be the displacements, in the directions of the axes, of one of the ions; then ex, ey, ez will be the components of the electric moment of a molecule, and, if a horizontal bar over the letters is employed to indicate mean values taken for a large number of particles, the components of the electric moment per unit volume will be

$$M_x = Ne\bar{x}, \quad M_y = Ne\bar{y}, \quad M_z = Ne\bar{z},$$

If the ions are in a state of vibration, they will excite in the aether a certain periodic dielectric displacement and a similar magnetic force; besides these, there may exist, independently of the ions, a disturbance of the equilibrium in the aether, in which there is a dielectric displacement, say (f_0, g_0, h_0).

Now, in order to obtain the equations of motion for one of the ions, I conceived a sphere B, whose radius, though very small in comparison with the wave-length, is very much larger than the

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[1]) The sign "etc." will always be used to indicate two equations similar to the one that is written down and relating to the axes of y and z.

[2]) LORENTZ, La théorie électromagnétique de MAXWELL et son application aux corps mouvants, Leiden, BRILL 1892. Also in Arch. néerl. T. 25.

molecular distances, and the centre of which is occupied by the molecule to be considered. I denoted by

$$\mathfrak{X}', \quad \mathfrak{Y}', \quad \mathfrak{Z}'$$

the components of the electric force at the centre of the sphere, in so far as it is due to the molecules within the surface, by

$$-\mathfrak{f}\,\mathrm{x}, \quad -\mathfrak{f}\,\mathrm{y}, \quad -\mathfrak{f}\,\mathrm{z}$$

the components of the elastic force by which the ion is driven back to its position of equilibrium, and by

$$\mathfrak{M}_x\,, \quad \mathfrak{M}_y\,, \quad \mathfrak{M}_z$$

three auxiliary functions, satisfying the equations

$$\left.\begin{aligned}\left(\Delta - \frac{1}{\mathrm{v}^2}\frac{\partial^2}{\partial t^2}\right)\mathfrak{M}_x &= -4\,\pi\,\mathrm{M}_x\,,\\ \left(\Delta - \frac{1}{\mathrm{v}^2}\frac{\partial^2}{\partial t^2}\right)\mathfrak{M}_y &= -4\,\pi\,\mathrm{M}_y\,,\\ \left(\Delta - \frac{1}{\mathrm{v}^2}\frac{\partial^2}{\partial t^2}\right)\mathfrak{M}_z &= -4\,\pi\,\mathrm{M}_z\,.\end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

In these the velocity of light in the aether is again represented by v. Finally, I found for the first of the three equations of motion [1])

$$\varkappa\frac{d^2\mathrm{x}}{dt^2} = -\,\mathfrak{f}\,\mathrm{x} + \frac{e^2}{\mathrm{v}}\frac{d^3\mathrm{x}}{dt^3} + \frac{4}{3}\,\pi\,\mathrm{v}^2\,e\,\mathrm{M}_x +$$
$$+\,\mathrm{v}^2\,e\left[\frac{\partial^2\mathfrak{M}_x}{\partial x^2} + \frac{\partial^2\mathfrak{M}_y}{\partial x\,\partial y} + \frac{\partial^2\mathfrak{M}_z}{\partial x\,\partial z} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\mathfrak{M}_x}{\partial t^2}\right] + 4\,\pi\,\mathrm{v}^2\,e\,f_0 + e\,\mathfrak{X}' \quad . \quad . \quad (6)$$

The term

$$\frac{e^2}{\mathrm{v}}\frac{d^3\mathrm{x}}{dt^3},$$

which corresponds to the damping of the vibrations by radiation, was shown to be so small that it may be neglected.

If, in (6), we replace some of the terms by their mean values, we shall find after division by e and after replacing $\mathfrak{X}'$ by M_x, multiplied by a constant,

[1]) l. c., § 128.

$$\frac{\mathrm{v}}{q}\,\mathrm{M}_x + \frac{\varkappa}{Ne^2}\frac{\partial^2 \mathrm{M}_x}{\partial t^2} =$$

$$= \mathrm{v}^2 \left[\frac{\partial^2 \mathfrak{M}_x}{\partial x^2} + \frac{\partial^2 \mathfrak{M}_y}{\partial x\, \partial y} + \frac{\partial^2 \mathfrak{M}_z}{\partial x\, \partial z} - \frac{1}{\mathrm{v}^2}\,\frac{\partial^2 \mathfrak{M}_x}{\partial t^2}\right] + 4\pi\,\mathrm{v}^2 f_0\,, \text{ etc. } \quad (7)$$

where q is a constant coefficient.

If the ion experiences a resistance, proportional to the velocity, we must introduce a term

$$-\,c\,\frac{d\mathrm{x}}{d\,t}$$

on the right-hand side of (6), and in the case of a magnetic field with the magnetic force (A, B, C), there will be a term

$$e\left(C\frac{d\mathrm{y}}{d\,t} - B\,\frac{d\mathrm{z}}{d\,t}\right).$$

Hence, the equations (7) will change into

$$\left.\begin{aligned} &\frac{\mathrm{v}}{q}\,\mathrm{M}_x + \frac{\varkappa}{Ne^2}\,\frac{\partial^2 \mathrm{M}_x}{\partial\, t^2} + \frac{c}{N e^2}\,\frac{\partial\, \mathrm{M}_x}{\partial\, t} - \frac{1}{N e}\left(C\frac{\partial\, \mathrm{M}_y}{\partial\, t} - B\,\frac{\partial\, \mathrm{M}_z}{\partial\, t}\right) = \\ &\quad = \mathrm{v}^2 \left[\frac{\partial^2 \mathfrak{M}_x}{\partial\, x^2} + \frac{\partial^2 \mathfrak{M}_y}{\partial x\, \partial y} + \frac{\partial^2 \mathfrak{M}_z}{\partial x\, \partial z} - \frac{1}{\mathrm{v}^2}\,\frac{\partial^2 \mathfrak{M}_x}{\partial\, t^2}\right] + 4\,\pi\,\mathrm{v}^2\, f_0\,, \end{aligned}\right\} \quad (8)$$

etc.

§ 4. The above formulae may be put in a form better known in the theory of electro-magnetism and admitting of direct comparison with the equations of VOIGT. We shall arrive at it, if we observe that there will be a magnetic force $\mathfrak{H}$, which may be decomposed into two parts, the one $\mathfrak{H}_1$ being produced by the vibrations of the ions, and the other $\mathfrak{H}_2$ belonging to the same state of motion as (f_0, g_0, h_0).

The components of the first of these parts are found to be [1])

$$\mathfrak{H}_{1x} = \frac{\partial^2 \mathfrak{M}_z}{\partial y \partial t} - \frac{\partial^2 \mathfrak{M}_y}{\partial z \partial t}\,,\ \mathfrak{H}_{1y} = \frac{\partial^2 \mathfrak{M}_x}{\partial z \partial t} - \frac{\partial^2 \mathfrak{M}_z}{\partial x \partial t}\,,\ \mathfrak{H}_{1z} = \frac{\partial^2 \mathfrak{M}_y}{\partial x \partial t} - \frac{\partial^2 \mathfrak{M}_x}{\partial y \partial t}\,, \quad (9)$$

and those of $\mathfrak{H}_2$ satisfy the equations

[1]) l. c., § 124.

$$\frac{\partial \mathfrak{H}_{2x}}{\partial t}=4\pi v^2\left(\frac{\partial g_0}{\partial z}-\frac{\partial h_0}{\partial y}\right),\ \frac{\partial \mathfrak{H}_{2y}}{\partial t}=4\pi v^2\left(\frac{\partial h_0}{\partial x}-\frac{\partial f_0}{\partial z}\right),$$

$$\frac{\partial \mathfrak{H}_{2z}}{\partial t}=4\pi v^2\left(\frac{\partial f_0}{\partial y}-\frac{\partial g_0}{\partial x}\right). \quad . \quad . \quad . \quad . \qquad (10)$$

Now, if we put

$$\mathfrak{E}_x = v^2\left[\frac{\partial^2 \mathfrak{M}_x}{\partial x^2}+\frac{\partial^2 \mathfrak{M}_y}{\partial x\,\partial y}+\frac{\partial^2 \mathfrak{M}_z}{\partial x\,\partial z}-\frac{1}{v^2}\,\frac{\partial^2 \mathfrak{M}_x}{\partial t^2}\right]+4\pi v^2 f_0, \text{ etc. } (11)$$

we shall find

$$\frac{\partial \mathfrak{E}_y}{\partial z}-\frac{\partial \mathfrak{E}_z}{\partial y}=\frac{\partial^2}{\partial t^2}\left[\frac{\partial \mathfrak{M}_z}{\partial y}-\frac{\partial \mathfrak{M}_y}{\partial z}\right]+4\pi v^2\left(\frac{\partial g_0}{\partial z}-\frac{\partial h_0}{\partial y}\right), \text{ etc.},$$

or, by (9) and (10),

$$\frac{\partial \mathfrak{E}_y}{\partial z}-\frac{\partial \mathfrak{E}_z}{\partial y}=\frac{\partial \mathfrak{H}_x}{\partial t},\ \frac{\partial \mathfrak{E}_z}{\partial x}-\frac{\partial \mathfrak{E}_x}{\partial z}=\frac{\partial \mathfrak{H}_y}{\partial t},\ \frac{\partial \mathfrak{E}_x}{\partial y}-\frac{\partial \mathfrak{E}_y}{\partial x}=\frac{\partial \mathfrak{H}_z}{\partial t}. \quad (3')$$

The form of these equations shows that $\mathfrak{E}$ is precisely the "electric force."

In virtue of (11) the equations (8) become

$$\frac{v}{q}\,M_x+\frac{\varkappa}{N e^2}\,\frac{\partial^2 M_x}{\partial t^2}+\frac{c}{N e^2}\,\frac{\partial M_x}{\partial t}-\frac{1}{N e}\left(C\frac{\partial M_y}{\partial t}-B\frac{\partial M_z}{\partial t}\right)=\mathfrak{E}_x, \text{ etc. } (4')$$

As we see, they express the relation between the electric force and the electric moment.

Finally, from (9) we may deduce the formula

$$\frac{\partial \mathfrak{H}_{1z}}{\partial y}-\frac{\partial \mathfrak{H}_{1y}}{\partial z}=\frac{\partial}{\partial t}\left[\frac{\partial^2 \mathfrak{M}_x}{\partial x^2}+\frac{\partial^2 \mathfrak{M}_y}{\partial x\,\partial y}+\frac{\partial^2 \mathfrak{M}_z}{\partial x\,\partial z}-\Delta\,\mathfrak{M}_x\right],$$

or, if (11) be taken into account,

$$\frac{\partial \mathfrak{H}_{1z}}{\partial y}-\frac{\partial \mathfrak{H}_{1y}}{\partial z}=\frac{\partial}{\partial t}\left[\frac{1}{v^2}\,\frac{\partial^2 \mathfrak{M}_x}{\partial t^2}-\Delta\,\mathfrak{M}_x\right]+\frac{1}{v^2}\,\frac{\partial \mathfrak{E}_x}{\partial t}-4\pi\frac{\partial f_0}{\partial t}.$$

Combining this with

$$4\pi\frac{\partial f_0}{\partial t}=\frac{\partial \mathfrak{H}_{2z}}{\partial y}-\frac{\partial \mathfrak{H}_{2y}}{\partial z},$$

5*

and attending to (5), we see that

$$\frac{\partial \mathfrak{H}_z}{\partial y}-\frac{\partial \mathfrak{H}_y}{\partial z}=4\pi\frac{\partial M_x}{\partial t}+\frac{1}{V^2}\frac{\partial \mathfrak{E}_x}{\partial t}.$$

Now, let a new vector $\mathfrak{D}$ be defined by the equation

$$\mathfrak{D}=M+\frac{\mathfrak{E}}{4\pi V^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

then:

$$\frac{\partial \mathfrak{H}_z}{\partial y}-\frac{\partial \mathfrak{H}_y}{\partial z}=4\pi\frac{\partial \mathfrak{D}_x}{\partial t},\quad \frac{\partial \mathfrak{H}_x}{\partial z}-\frac{\partial \mathfrak{H}_z}{\partial x}=4\pi\frac{\partial \mathfrak{D}_y}{\partial t},$$

$$\frac{\partial \mathfrak{H}_y}{\partial x}-\frac{\partial \mathfrak{H}_x}{\partial y}=4\pi\frac{\partial \mathfrak{D}_z}{\partial t} \quad . \quad . \quad . \quad (1')$$

Since $\mathfrak{E}$ is the electric force, $\mathfrak{E}/4\pi V^2$ will be the dielectric displacement in the aether; $\mathfrak{D}$ will therefore be the total dielectric displacement and $\dot{\mathfrak{D}}$ the displacement-current. Thus the equations (1′) are seen to contain the well known relation between the magnetic force and the electric current.

In (1′), (2′), (3′) and (4′) we have got the complete system of equations of motion. We might have obtained them also by starting from the relation between $\mathfrak{E}$ and M, which I have assumed in my „Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern"; it would only have been necessary to add the terms which arise from a resistance and from the action of a magnetic field. The above method is less simple, but it goes farther in explaining the mechanism of the phenomena.

§ 5. Now, it is easily seen that the equations of the electromagnetic theory are identical with those of VOIGT, if in these latter only one vector P is assumed.

Indeed, if, in the formulae of VOIGT, we replace

$$\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t},\ \xi,\ \eta,\ \zeta,\ U,\ V,\ W,\ \Xi,\ H,\ Z$$

by

$$\frac{\mathfrak{H}_x}{4\pi V^2},\ \frac{\mathfrak{H}_y}{4\pi V^2},\ \frac{\mathfrak{H}_z}{4\pi V^2},\ \frac{\mathfrak{D}_x}{V^2},\ \frac{\mathfrak{D}_y}{V^2},\ \frac{\mathfrak{D}_z}{V^2},$$

$$-\frac{M_x}{\varepsilon},\ -\frac{M_y}{\varepsilon},\ -\frac{M_z}{\varepsilon},\ \frac{\mathfrak{E}_x}{4\pi V^2},\ \frac{\mathfrak{E}_y}{4\pi V^2},\ \frac{\mathfrak{E}_z}{4\pi V^2},$$

the equations (2) and (3) change into (2′) and (3′), and the formulae (1), if first differentiated with regard to t, take the form (1′).

As to the equations of motion (4), these become

$$-\frac{1}{\varepsilon}\frac{\partial^2 \mathrm{M}_x}{\partial t^2}-\frac{d}{\varepsilon}\mathrm{M}_x-\frac{f}{\varepsilon}\frac{\partial \mathrm{M}_x}{\partial t}-\frac{g}{\varepsilon}\left(B\frac{\partial \mathrm{M}_z}{\partial t}-C\frac{\partial \mathrm{M}_y}{\partial t}\right)+\frac{\varepsilon \mathfrak{D}_x}{\mathrm{v}^2}=0, \text{etc.} \quad (12)$$

or, if we put for $\mathfrak{D}$ the value (2′), and multiply by $\dfrac{4\pi \mathrm{v}^4}{\varepsilon}$,

$$4\pi \mathrm{v}^2\left(\frac{\mathrm{v}^2 d}{\varepsilon^2}-1\right)\mathrm{M}_x+\frac{4\pi \mathrm{v}^4}{\varepsilon^2}\frac{\partial^2 \mathrm{M}_x}{\partial t^2}+$$

$$+\frac{4\pi \mathrm{v}^4 f}{\varepsilon^2}\frac{\partial \mathrm{M}_x}{\partial t}-\frac{4\pi \mathrm{v}^4 g}{\varepsilon^2}\left(C\frac{\partial \mathrm{M}_y}{\partial t}-B\frac{\partial \mathrm{M}_z}{\partial t}\right)=\mathfrak{E}_x, \text{ etc.} \quad . \; . \; (13)$$

This agrees with (4′)[1]. At the same time we are led to the following relations between the coefficients

$$4\pi \mathrm{v}\left(\frac{\mathrm{v}^2 d}{\varepsilon^2}-1\right)=\frac{1}{q}, \quad \frac{4\pi \mathrm{v}^4}{\varepsilon^2}=\frac{\varkappa}{Ne^2},$$

$$\frac{4\pi \mathrm{v}^4 f}{\varepsilon^2}=\frac{c}{Ne^2}, \quad \frac{4\pi \mathrm{v}^4 g}{\varepsilon^2}=\frac{1}{Ne}. \quad . \; . \; . \; . \; . \; . \; (14)$$

§ 6. If we suppose a molecule to contain a certain number of ions, each of which can be displaced from its position of equilibrium, the total electric moment M may be split up into the parts $\mathrm{M}_1, \mathrm{M}_2, \ldots$, corresponding to the displacements of the separate ions. In this case, the equations (1′), (2′) and (3′) will still hold, but instead of (4′) we shall have as many times three equations, as there are ions in the molecule. For the sake of brevity, I shall put $\mathfrak{X}'=\mathfrak{Y}'=\mathfrak{Z}'=0$[2]).

If, now, we wish to write down the equation (6) for the h^{th} ion, we have to replace x by x_h, but the term $\frac{4}{3}\pi \mathrm{v}^2 e \mathrm{M}_x$ will still contain the total moment M_x. Instead of (4′) we shall therefore get

[1]) Voigt's formulae in Wied. Ann., Bd. 67, p. 345 are likewise of the same form.
[2]) l. c. § 105.

$$\left.\begin{aligned}&\frac{f_h}{\Lambda e_h^2}\mathrm{M}_{hx}-\frac{4}{3}\pi \mathrm{v}^2\mathrm{M}_x+\frac{\varkappa_h}{Ne_h^2}\frac{\partial^2\mathrm{M}_{hx}}{\partial t^2}+\\&\qquad+\frac{c_h}{Ne_h^2}\frac{\partial \mathrm{M}_{hx}}{\partial t}-\frac{1}{Ne_h}\left(C\frac{\partial \mathrm{M}_{hy}}{\partial t}-B\frac{\partial \mathrm{M}_{hz}}{\partial t}\right)=\mathfrak{E}_x,\ \text{etc.}\end{aligned}\right\}\ldots (4'')$$

The equations (1), (2) and (3) of VOIGT, taken this time in their general form, may again be written in the form of (1'), (2') and (3'); for this purpose it will suffice to replace as before

$$\frac{\partial u}{\partial t},\quad \xi,\quad \Xi,\ \text{etc.}$$

by

$$\frac{\mathfrak{H}_x}{4\pi \mathrm{v}^2},\quad \frac{\mathfrak{D}_x}{\mathrm{v}^2},\quad \frac{\mathfrak{E}_x}{4\pi \mathrm{v}^2},\ \text{etc.}$$

and

$$U_h,\quad V_h,\quad W_h$$

by

$$-\frac{\mathrm{M}_{hx}}{\varepsilon_h},-\frac{\mathrm{M}_{hy}}{\varepsilon_h},-\frac{\mathrm{M}_{hz}}{\varepsilon_h}.$$

From (4) we shall get equations, similar to (12). They will however contain the indices h, and if we use the value

$$\mathfrak{D}=\mathrm{M}+\frac{\mathfrak{E}}{4\pi \mathrm{v}^2},$$

we obtain an equation which is only slightly different from (13). The first term in it will be

$$\frac{4\pi \mathrm{v}^4 d_h}{\varepsilon_h^2}\mathrm{M}_{hx}-4\pi \mathrm{v}^2\mathrm{M}_x,$$

instead of

$$4\pi \mathrm{v}^2\left(\frac{\mathrm{v}^2 d}{\varepsilon^2}-1\right)\mathrm{M}_x$$

in (13), and in the following terms M as well as the coefficients must be written with the index h.

Finally, by assuming similar relations between the coefficients as in § 5 above, this new equation will become nearly, but not quite identical with (4''), the difference consisting in this, that it will not contain

$$- \frac{4}{3} \pi \mathrm{v}^2 \mathrm{M}_x ,$$

but

$$- 4 \pi \mathrm{v}^2 \mathrm{M}_x .$$

For our purpose this is of no consequence. We shall confine ourselves to the case of molecules with a single movable ion or a single vector P, and even if we were to consider a more general case, our conclusions would not be materially altered.

§ 7. POINCARÉ investigates the propagation of plane waves in the direction of the axis of z. He introduces no resistance, but he assumes the existence of several ions in each molecule.

In his paper (X, Y, Z) denotes the total „dielectric polarization," (X_h, Y_h, Z_h) one of its parts, (f, g, h) the dielectric displacement. His equations, if written partly in the above notations, are

$$\lambda_h \frac{\partial^2 X_h}{\partial t^2} + \frac{X_h}{L_h} = f + \frac{1}{3} X + \varepsilon_h \left(\frac{\partial Y_h}{\partial t} C - \frac{\partial Z_h}{\partial t} B \right) \text{, etc.} \quad (15)$$

with the constants λ_h, L_h and ε_h, and [1])

$$\left. \begin{aligned} \frac{\partial^2 f}{\partial z^2} - \frac{1}{\mathrm{v}^2} \frac{\partial^2 f}{\partial t^2} &= \frac{1}{\mathrm{v}^2} \frac{\partial^2 X}{\partial t^2} , \\ \frac{\partial^2 g}{\partial z^2} - \frac{1}{\mathrm{v}^2} \frac{\partial^2 g}{\partial t^2} &= \frac{1}{\mathrm{v}^2} \frac{\partial^2 Y}{\partial t^2} , \\ h + Z &= 0 . \end{aligned} \right\} \quad (16)$$

Now, if $\mathfrak{H}$ and $\mathfrak{E}$ do not contain x and y, we shall have by our equations (1') and (3')

$$- \frac{\partial \mathfrak{H}_y}{\partial z} = 4 \pi \frac{\partial \mathfrak{D}_x}{\partial t} , \; \frac{\partial \mathfrak{H}_x}{\partial z} = 4 \pi \frac{\partial \mathfrak{D}_y}{\partial t} , \; 0 = \frac{\partial \mathfrak{D}_z}{\partial t} .$$

and

$$\frac{\partial \mathfrak{E}_y}{\partial z} = \frac{\partial \mathfrak{H}_x}{\partial t} , \quad - \frac{\partial \mathfrak{E}_x}{\partial z} = \frac{\partial \mathfrak{H}_y}{\partial t} , \; 0 = \mathfrak{H}_z .$$

[1]) By a typographical error, the formula of POINCARÉ which corresponds to the first two of the equations (16) has on the left-hand side the sign +.

Hence

$$\left.\begin{aligned}\frac{\partial^2 \mathfrak{E}_y}{\partial z^2} &= 4\pi \frac{\partial^2 \mathfrak{D}_y}{\partial t^2} ,\\ \frac{\partial^2 \mathfrak{E}_x}{\partial z^2} &= 4\pi \frac{\partial^2 \mathfrak{D}_x}{\partial t^2} ,\\ \mathfrak{D}_z &= 0 .\end{aligned}\right\} \quad \ldots\ldots\ldots (17)$$

Let the components of the dielectric displacement in the aether be f, g, h. Then

$$\mathfrak{E}_y = 4\pi \mathrm{v}^2 g , \qquad \mathfrak{E}_z = 4\pi \mathrm{v}^2 h ,$$

and if, instead of (M_x, M_y, M_z), we write (X, Y, Z) for the electric moment per unit of volume,

$$\mathfrak{D}_x = f + X , \qquad \mathfrak{D}_y = g + Y , \qquad \mathfrak{D}_z = h + Z .$$

Putting this in (17), we are led to the equations (16), which I have just taken from POINCARÉ.

Again, if there be no resistance, our equations (4′) may now be replaced by

$$\frac{\mathfrak{f}_h}{N e_h^2} X_h - \frac{4}{3}\pi \mathrm{v}^2 X + \frac{\varkappa_h}{N e_h^2}\frac{\partial^2 X_h}{\partial t^2} -$$

$$- \frac{1}{N e_h}\left(C\frac{\partial Y_h}{\partial t} - B\frac{\partial Z_h}{\partial t}\right) = 4\pi \mathrm{v}^2 f , \quad \text{etc.}$$

Dividing this by $4\pi \mathrm{v}^2$, and putting

$$\frac{\varkappa_h}{4\pi \mathrm{v}^2 N e_h^2} = \lambda_h ,$$

$$\frac{\mathfrak{f}_h}{4\pi \mathrm{v}^2 N e_h^2} = \frac{1}{L_h} ,$$

$$\frac{1}{4\pi \mathrm{v}^2 N e_h} = \varepsilon_h ,$$

we find the above formulae (15). POINCARÉ's equations are thus found to be identical with mine.

§ 8. In the application of the equations to the phenomena in question, I shall follow VOIGT's treatment of the case of a single vector P.

In the first place VOIGT examines the propagation of light along the lines of force, which are supposed parallel to the axis of x.

He denotes

by R the strength of the field,

by ϑ the time of vibration, divided by 2π,

by ω the velocity of propagation, and by $\varkappa$ the coefficient of absorption, in this sense, that over a distance equal to the wave-length the amplitude is diminished in ratio of 1 to $e^{-2\pi\varkappa}$.

Further he puts:

$$\varepsilon^2/d = q^2\,\mathrm{v}^2,\quad {}^1/d = \vartheta_0^{\,2},\quad f/d = \vartheta',\quad g/d = k\ ^{1)}\ .\ \ .\quad (18)$$

The values of ω and $\varkappa$ for circularly polarized light are given by VOIGT's formulae (24) and (25), in which the upper signs are to be taken if the polarization is right-handed, and the lower signs if it is left-handed. To simplify these formulae, I shall put

$$\vartheta^2 \pm k\,R\,\vartheta - \vartheta_0^{\,2} = S;$$

we have then

$$\frac{\omega^2(1-\varkappa^2)}{(1+\varkappa^2)^2} = \mathrm{v}^2\left(1 - \frac{q^2\,\vartheta^2\,S}{S^2 + \vartheta'^2\,\vartheta^2}\right),$$

$$\frac{2\,\omega^2\,\varkappa}{(1+\varkappa^2)^2} = \frac{\mathrm{v}^2\,q^2\,\vartheta'\,\vartheta^3}{S^2 + \vartheta'^2\,\vartheta^2}.$$

Now, we may suppose that even the maximum value of $\varkappa$ is a very small fraction. The left hand members of the equations may therefore be written

$$\omega^2 \text{ and } 2\,\omega^2\,\varkappa;$$

hence, by division,

$$2\,\varkappa = \frac{q^2\,\vartheta'\,\vartheta^3}{S^2 - q^2\,\vartheta^2\,S + \vartheta'^2\,\vartheta^2}\ .\ \ .\ \ .\ \ .\ \ .\quad (19)$$

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[1]) $2\,\pi\,\vartheta_0$ is the period of the free vibrations of which the ions are capable. As to the time ϑ', it depends on the magnitude of the resistance.

Our next question is, for what value of ϑ this will be a maximum. At all events this value will lie in the neighbourhood of ϑ_0, and if the absorption bands are narrow, it will be permitted to replace ϑ by ϑ_0 in the numerator of (19). Consequently, the denominator, for which we may write

$$\left(S - \frac{1}{2} q^2 \vartheta^2\right)^2 + \vartheta'^2 \vartheta^2 - \frac{1}{4} q^4 \vartheta^4 \quad . \quad . \quad . \quad . \quad (20)$$

must become a minimum. I shall neglect the variation of the two last terms, and replace in them ϑ by ϑ_0. Then, the minimum will be reached if

$$S = \tfrac{1}{2} q^2 \vartheta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and the maximum of absorption will be determined by

$$2\,\varkappa_{\text{max.}} = \frac{q^2 \vartheta' \vartheta_0^3}{\vartheta'^2 \vartheta_0^2 - \frac{1}{4} q^4 \vartheta_0^4}.$$

In order that this may be very small, I shall suppose that q^2 is greatly inferior to $\frac{\vartheta'}{\vartheta_0}$. In this case, the last term in the denominator may be neglected, so that

$$2\,\varkappa_{\text{max.}} = q^2 \frac{\vartheta_0}{\vartheta'}.$$

At the same time we see that the last term in (20) may be neglected in comparison with the preceding one; consequently our result will be true, provided that we may neglect the variation of the second term in (20), while the first term passes through its minimum. This condition will always be fulfilled, if the absorption bands are sufficiently narrow.

The equation (21) may be replaced by

$$\vartheta^2 \pm k R \vartheta - \vartheta_0^2 = \tfrac{1}{2} q^2 \vartheta_0^2.$$

We shall suppose ϑ' much smaller than ϑ_0. Then, from what has been said, q^2 will be much smaller than 1, and in the absence of a magnetic field, i. e. for $R = 0$, the maximum of absorption will lie in the immediate neighbourhood of ϑ_0. If, moreover, $kR\vartheta_0$ be very large in comparison with $\frac{1}{2} q^2 \vartheta_0^2$, we shall have approximately

$$\vartheta^2 \pm k R \vartheta - \vartheta_0^2 = 0,$$

or

$$\vartheta = \vartheta_0 \mp \tfrac{1}{2} k R,$$

since $k R$ must be small with regard to ϑ_0.

Now, in order that a distinct doublet may be seen, the distance of the components must be large as compared with the breadth of the absorption bands.

Replacing (19) by

$$2 \varkappa = \frac{q^2 \vartheta' \vartheta_0^3}{(S - \frac{1}{2} q^2 \vartheta^2)^2 + \vartheta'^2 \vartheta_0^2},$$

we see at once that for a value of ϑ, such that

$$S - \tfrac{1}{2} q^2 \vartheta^2 = \pm \mu \vartheta' \vartheta_0$$

the value of $\varkappa$ will be

$$\frac{\varkappa_{max}}{1 + \mu^2}.$$

We may therefore consider the *borders* of the absorption band to be determined by the last equation, if in it we take for μ a moderate number, say 5. Hence, the necessary condition for a distinct doublet is seen to be $k R > \mu \vartheta'$. If it is fulfilled, our above supposition as to the value of $k R \vartheta_0$ will likewise hold good. Indeed, we shall have

$$k R \vartheta_0 > \mu \vartheta' \vartheta_0,$$

whereas $q^2 \vartheta_0^2$ is much smaller than $\vartheta' \vartheta_0$.

§ 9. We now come to the propagation of light in a direction perpendicular to the lines of force. Let the vectors P be also perpendicular to these lines, i. e. in the language of the electromagnetic theory of light, let the electric vibrations take place at a right angle to the direction of the field. Then, according to VOIGT, the velocity of propagation ω, and the absorption $\varkappa$ will be determined by his formulae (50) and (51), or if we neglect $\varkappa^2$, by

$$\omega^2 = \mathrm{v}^2 \left[1 - \frac{\frac{1}{2} q^2 \vartheta^2 S_1}{S_1^2 + \vartheta'^2 \vartheta^2} - \frac{\frac{1}{2} q^2 \vartheta^2 S_2}{S_2^2 + \vartheta'^2 \vartheta^2}\right] \quad . \quad . \quad . \quad (22)$$

and

$$2\,\omega^2\,\varkappa = \tfrac{1}{2}\,v^2\,q^2\,\vartheta'\,\vartheta^3\left[\frac{1}{S_1^2+\vartheta'^2\,\vartheta^2}+\frac{1}{S_2^2+\vartheta'^2\,\vartheta^2}\right] \quad . \quad (23)$$

Here I have put

$$\vartheta^2 - k\,R\,\vartheta - \vartheta_0^2 = S_1$$

and

$$\vartheta^2 + k\,R\,\vartheta - \vartheta_0^2 = S_2.$$

It is easily seen that, with the assumptions we have made concerning the magnitude of the different terms, the equations (22) and (23) imply the existence of *two* absorption-bands, corresponding to

$$S_1 = 0 \quad \text{and} \quad S_2 = 0.$$

These bands are precisely the outer components of the triplet, one is led to by the elementary theory of the ZEEMAN-effect.

The breadth of each of these lines will be equal to that of the original absorption-band; in virtue of our assumptions it will be much smaller than the distance of the two lines.

Now it is clear that such a thing would be impossible, if the modification of the propagation of light were so small as POINCARÉ finds, namely of the order of R^2, if R is the strength of the field. If, by the action of the magnetic field, the maximum of absorption is shifted to a place, where the absorption was originally insensible, we have to do with a finite change at this point of the spectrum.

§ 10. In order to examine this more closely, we must return to the equations of motion themselves, from which the formulae (22) and (23) have been deduced. Let the magnetic force be parallel to the axis of z ($A = B = 0$, $C = R$) and let the propagation of light take place along the axis of x. Then the complex expressions, which satisfy the equations of motion and whose real parts are the values of U, V, W, ξ, η, ζ, etc., will contain the common factor

$$e^{-\frac{\varkappa x}{\vartheta\omega} + i\frac{1}{\vartheta}\left(t - \frac{x}{\omega}\right)}.$$

There will arise no confusion, if we use the letters U, V, etc. themselves to represent these complex expressions.

Let the vector P be perpendicular to the axis of z. Then

$$W = 0, \quad \text{and} \quad Z = 0.$$

By the equations (3) we find:

$$u = 0, \quad v = 0, \quad w = -\frac{\vartheta}{\omega}(\varkappa + i)\,\Pi,$$

by (1):

$$\xi = 0, \quad \eta = -\frac{1}{\omega^2}(\varkappa + i)^2\,\Pi, \quad \zeta = 0,$$

and by (2):

$$\Xi = \varepsilon U, \quad H = -\frac{\mathrm{v}^2}{\omega^2}(\varkappa + i)^2\,\Pi + \varepsilon V.$$

Hence

$$\Pi = \frac{\varepsilon V}{1 + \frac{\mathrm{v}^2}{\omega^2}(\varkappa + i)^2}$$

and

$$\eta = -\frac{(\varkappa + i)^2}{\omega^2 + \mathrm{v}^2(\varkappa + i)^2}\,\varepsilon V.$$

Consequently, the two first of the equations (4) become

$$\left(-\frac{1}{\vartheta^2} + d + \frac{if}{\vartheta}\right)U - \frac{igR}{\vartheta}V = 0$$

and

$$\left(-\frac{1}{\vartheta^2} + d + \frac{if}{\vartheta}\right)V + \frac{igR}{\vartheta}U - \frac{(\varkappa + i)^2}{\omega^2 + \mathrm{v}^2(\varkappa + i)^2}\varepsilon^2 V = 0,$$

or, if we introduce the quantities ϑ_0, ϑ', etc.,

$$(\vartheta^2 + i\vartheta\vartheta' - \vartheta_0^2)\,U - ikR\vartheta V = 0 \quad . \quad . \quad . \quad (24)$$

$$(\vartheta^2 + i\vartheta\vartheta' - \vartheta_0^2)\,V + ikR\vartheta U - \frac{(\varkappa + i)^2}{\omega^2 + \mathrm{v}^2(\varkappa + i)^2}q^2\vartheta^2\mathrm{v}^2 V = 0 \quad (25)$$

These equations correspond to the two last of POINCARÉ's formulae (6), and if we were to follow his mode of reasoning, we should say that, in virtue of (24), U must be a small quantity of the order R so that the second term in (25) becomes of the order of R^2. We should then omit this last term; all influence of the magnetic field would thereby disappear from (25).

There is however an error in this reasoning, because, as I shall now show, the coefficient of U in (24) may become of the same order as that of V.

We saw already that the place of the absorption-lines is determined by the conditions

$$S_1 = 0 \quad \text{and} \quad S_2 = 0 ,$$

i. e. by

$$\vartheta^2 - \vartheta_0^2 = \pm k R \vartheta .$$

We have further assumed that $k R \vartheta$ is much larger than $\vartheta \vartheta'$ Hence, in the equation (24), the coefficient of U is approximately $\pm k R \vartheta$, so that

$$U = \pm i V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

On the other hand, we may neglect in (25) the last term, at least if ϑ has a value for which the absorption is a maximum. For, neglecting $\varkappa^2$, we find for that term

$$\frac{1 - 2 i \varkappa}{(\omega^2 - \mathrm{v}^2) + 2 \mathrm{v}^2 i \varkappa} q^2 \vartheta^2 \mathrm{v}^2 V . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The equations (22) and (23) show that, in the middle of one of the absorption-bands, $\omega^2 - \mathrm{v}^2$ is much smaller than $2 \mathrm{v}^2 \varkappa$. We may therefore neglect the first term in the denominator. Omitting likewise in the numerator the second term, which by our assumption, lies far beneath 1, we find for (27)

$$\frac{q^2 \vartheta_0^2 \mathrm{v}^2 V}{2 \mathrm{v}^2 i \varkappa} = - \frac{i q^2 \vartheta_0^2}{2 \varkappa} V ,$$

But, according to (23), the maximal absorption is given by

$$2 \varkappa = \tfrac{1}{2} \frac{q^2 \vartheta_0}{\vartheta'} ;$$

(27) may therefore be finally replaced by

$$- 2 i \vartheta' \vartheta_0 V ,$$

a quantity that may be omitted in (25) as well as $i \vartheta \vartheta' V$ in the first term of that equation. In this way (25) reduces to

$$V = \mp i U ,$$

which agrees with (26).

Translated into the terms of the electromagnetic theory of light, our result becomes

$$M_x = \pm i\, M_y\,.$$

The meaning of this is that the ions move in circles perpendicular to the lines of force, the direction of this motion being opposite in the two cases, represented by the two outer lines of the absorption-triplet.

The assumptions we have found necessary in the foregoing considerations, viz. that the inequalities

$$q^2 < \frac{\vartheta'}{\vartheta_0} \quad \text{and} \quad k R > \vartheta'$$

exist in a high degree, imply that kR is much larger than $q^2 \vartheta_0$.

If this is to be the case, it follows from (18) that $\frac{g}{c^2} R\, v^2$ must far surpass ϑ_0; in the language of the electromagnetic theory of light this means — as may be seen from (14) — that

$$\frac{R}{4 \pi v^2 N e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

must largely exceed the time ϑ_0.

Of course this condition can always be fulfilled if only the number of molecules N can be made small enough. This was to be expected because at very small densities the molecules must become independent of one another, and this is precisely what is assumed in the elementary theory.

It would be difficult to state accurately at what density the expression (28) becomes so large that distinct triplets may be seen. The preceding considerations show however that the triplets must appear in all cases where the observations along the lines of force give a good doublet.

Astronomy. — "*On the finding back of the Comet of* HOLMES *according to the computations of* Mr. H. J. ZWIERS." By Prof. H. G. VAN DE SANDE BAKHUYZEN.

In the Transactions of the Royal Academy of Sciences 1st Section Vol. III appeared a paper by Mr. H. J. ZWIERS on the orbit of the comet of HOLMES, observed from Nov. 8th 1892 till March 13th 1893. From these observations Mr. ZWIERS has deduced with great care the most probable orbit, which proved to be an ellipse, in which the comet at its greatest distance from the sun approaches