Chemistry. — *"On Isomorphous Compounds of Gold and Mercury."* By Prof. Th. H. Behrens.

In his Manual of Microchemical Analysis the auther has pointed to analogies between thiocyanates of gold and mercury.

Renewed investigation of this subject has shown, that the isomorphism of these double thiocyanates cannot be fully established by means of compound crystals. Halogen compounds have then been tried, and from these complete series of compound crystals have been obtained. They were prepared by adding to mixed solutions of the chlorides and bromides of gold and mercury chlorides or bromides of thallium, caesium and rubidium. Thallous compounds act promptly; the compound crystals are interspersed with flakes of trichloride or tribromide of thallium. The action of caesium and rubidium compounds is slower and less energetic. It can be hastened and furthered by adding about one tenth~part of alcohol. This takes up one third of the halogen, that was combined with gold (shown by a change of colour in the solution of bromides) while gold dichloride or — dibromide is fixed in the compound crystals along with dichloride or dibromide of mercury. If no alcohol is added the halogen, split off from the gold trihaloid must form trihaloid of caesium or rubidium, which is also readily attacked by hydrolysis.

Finally it may be mentioned, that the compound crystals of bromides will be found useful in testing for gold. With caesium the solubility is small, and the yellow colour of the crystals is seen without difficulty with a proportion of one part of gold to fifty parts of mercury.

Physics. — Prof. J. D. van der Waals presents on behalf of Dr. G. Bakker of Schiedam a paper on: *"A remark on the Molecular Potential Function of Prof. van der Waals."*

In his „Thermodynamische Theorie der Capillariteit in de onderstelling van continue dichtheidsverandering" Prof. van der Waals finds for the potential of two material points at a distance r the expression

$$P = C - f \frac{e^{-\frac{r}{\lambda}}}{r}$$

in which C, f and λ represent the constants.

Some time later[1]) Prof. VAN DER WAALS pointed out a remarkable property of that function. He found that if a coefficient depending on the radius, is left out of account, in consequence of this function the potential of a homogeneous sphere for an exterior point is determined by the distance between the point and the centre of the sphere in the same way as if the whole mass were concentrated in the centre.

On account of the great importance, practical as well as theoretical, of such a function for a theory of gases and liquids, which assumes spherical molecules (by which the potential energy might de determineb in a simple way by the configuration of the centres of the molecules), I examined the question whether there are more potential functions, which possess this property. As a solution I found the general function:

$$\varphi(r) = \frac{A e^{-qr}}{r} + \frac{B e^{qr}}{r} + C \ . \ . \ . \ . \ . \ . \ (1)$$

in which A and B are arbitrary positive and negative constants. For a spherical shell the coefficient depends on the radius in the following manner:

$$F(R) = \frac{e^{q R} - e^{-q R}}{2 q R} \ .$$

If however, we restrict ourselves to *attractive* forces, which decrease according to the distance, the *most general* function is that of VAN DER WAALS, viz:

$$\varphi(r) = C - \frac{f e^{-qr}}{r} \ .$$

If for this potential function a spherical (homogeneous) mass assumes this property, it will also be the case for a spherical shell and *vice versa*.
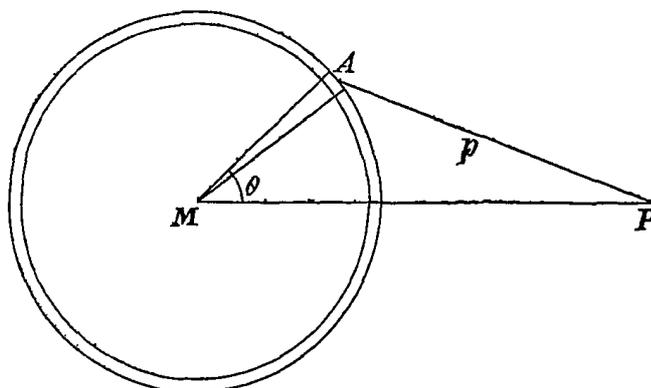

Fig. 1.

Let R be the radius of a spherical shell which is thought infinitely thin, P the point on which the shell acts, dR the thickness and M the centre of the shell.

Let us imagine a cone with an infinitely small aperture

[1]) See "Zeitschrift für physikalische Chemie", XIII, 4, Seite 720, 1894.

$d\omega$, of which M is the vertex, then this cone will cut from the shell a volume $R^2\, dR\, d\omega$. If ϱ is the density, the mass of the element in A is: $R^2\, dR\, d\omega\, \varrho$. If $\varphi(r)$ represents the form of the potential function, the potential energy of a unity of mass in P in consequence of the element in A is: $R^2\, dR\, d\omega\, \varrho\, \varphi(p)$, p representing the distance between A and P.

If we turn the figure round MP as axis, the element in A describes an annular space, so that $\int d\omega = 2\pi \sin\theta\, d\theta$; θ representing $\angle AMP$. In consequence of the annular space the potential energy in P is:

$$2\pi R^2\, dR\, \sin\theta\, d\theta\, \varrho\, \varphi(p) = -2\pi R^2\, dR\, d\cos\theta\, \varrho\, \varphi(p) \ .$$

Now $p^2 = R^2 + r^2 - 2Rr\cos\theta$, in which $r = MP$, so

$$2p\, dp = -2Rr\, d\cos\theta \ .$$

The expression for the potential energy becomes therefore:

$$\varrho\, \frac{2\pi R^2\, dR}{Rr}\, p\, dp\, \varphi(p)$$

or because $4\pi R^2\, dR\, \varrho$ represents the mass of the shell:

$$\frac{1}{2}\, \frac{M}{Rr}\, p\, \varphi(p)\, dp \ .$$

The integration over the whole of the shell gives:

$$\frac{1}{2}\, \frac{M}{Rr}\int_{r-R}^{r+R} p\, \varphi(p)\, dp \ .$$

If $F(R)$ represents the before mentioned coefficient and $F(R)$ a function of R which is also to be determined, we may write:

$$\frac{1}{2}\, \frac{M}{Rr}\int_{r-R}^{r+R} p\, \varphi(p)\, dp = F(R)\, M\, \varphi(r) + F(R)\, M \ ,$$

If we leave an *absolute constant* out of account, this equation furnishes the potential function, belonging to a force acting in the required manner.

Let us put:

$$\int_{r-R}^{r+R} p\,\varphi\,(p)\,dp = 2\,Rr\,\mathrm{F}\,(R)\,\varphi\,(r) + 2\,Rr\,F\,(R) \quad . \quad . \quad . \quad (a)$$

If we differentiate this identity twice with respect to r and also twice with respect to R, and put $\int r\,\varphi\,(r)\,dr = \psi\,(r)$, we get:

$$\psi''\,(r+R) - \psi''\,(r-R) = 4\,R\,\mathrm{F}\,(R)\,\varphi'\,(r) + 2\,Rr\,F\,(R)\,\varphi''\,(r)$$

and

$$\psi''\,(r+R) - \psi''\,(r-R) = 4\,r\,\varphi\,(r)\,\mathrm{F}'\,(R) + 2\,Rr\,\varphi\,(r)\,\mathrm{F}''\,(R) +$$
$$+ 4\,r\,F'\,(R) + 2\,Rr\,F''\,(R) \; .$$

The left side members of these equations are the same, so also

$$2\,R\,\mathrm{F}\,(R)\,\varphi'\,(r) + Rr\,\mathrm{F}\,(R)\,\varphi''\,(r) = 2\,r\,\varphi\,(r)\,\mathrm{F}'\,(R) +$$
$$+ Rr\,\varphi\,(r)\,\mathrm{F}''\,(R) + 2\,r\,F'\,(R) + Rr\,F''\,(R)$$

or

$$\frac{2\,\varphi'\,(r) + r\,\varphi''\,(r)}{r\,\varphi\,(r)} = \frac{2\,\mathrm{F}'\,(R) + R\,\mathrm{F}''\,(R)}{R\,\mathrm{F}\,(R)} + \frac{1}{\varphi\,(r)}\;\frac{2\,F'\,(R) + R\,F''\,(R)}{R\,\mathrm{F}\,(R)} \; .$$

R and r not being dependent on each other, we get separately

$$1 \;.\;.\;.\;.\;\; \frac{2\,\mathrm{F}'\,(R) + R\,\mathrm{F}''\,(R)}{R\,\mathrm{F}\,(R)} = C_1, \text{ in which } C_1 \text{ is an absolute constant}$$

$$2 \;.\;.\;.\;.\;\; \frac{2\,F'\,(R) + R\,F''\,(R)}{R\,\mathrm{F}\,(R)} = C_2 \text{ also absolutely constant}$$

$$\text{and } 3 \;.\;.\;.\;.\;\; \frac{2\,\varphi'\,(r) + r\,\varphi''\,(r)}{r\,\varphi\,(r)} = C_1' + \frac{C_2}{\varphi\,(r)} \; .$$

The solution of equation 3 will furnish the general form for the required potential function.

If we write $r = x$ and $\varphi\,(r) = y$, the last equation becomes

$$\frac{d^2 y}{dx^2} + \frac{2}{x}\,\frac{dy}{dx} - C_1\,y = C_2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$x\,\frac{d^2 y}{dx^2} + 2\,\frac{dy}{dx} = C_1\,xy + C_2\,x$$

$$\frac{d^2\left(xy + \dfrac{C_2}{C_1}\,x\right)}{dx^2} - C_1\left(xy + \frac{C_2}{C_1}\,x\right) = 0 \; .$$

This equation has, according to C_1 being positive or negative, the solutions:

$$xy + \frac{C_2}{C_1} x = Ae^{-x\sqrt{C_1}} + Be^{x\sqrt{C_1}}$$

or

$$xy + \frac{C_2}{C_1} x = A_1 \sin (x\sqrt{-C_1} + \alpha)$$

in which A, B, A_1 and α are arbitrary constants.

The potential function becomes therefore:

$$\varphi (r) = \frac{Ae^{-r\sqrt{C_1}} + Be^{r\sqrt{C_1}}}{r} - \frac{C_2}{C_1} \quad . \quad . \quad . \quad . \quad (5)$$

or

$$\varphi (r) = \frac{A_1 \sin (r\sqrt{-C_1} + \alpha)}{r} - \frac{C_2}{C_1} \quad . \quad . \quad . \quad . \quad (6)$$

If we put $C_1 = q^2$ in the first case and $C_1 = -q^2$ in the second case, the potential functions become:

$$\varphi (r) = \frac{Ae^{-qr} + Be^{qr}}{r} - \frac{C_2}{q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

or

$$\varphi (r) = \frac{A_1 \sin (qr + \alpha)}{r} - \frac{C_2}{q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

If we restrict ourselves to functions which relate to forces as they occur in nature, the second potential function must be excluded, and according to an above mentioned remark, the most general expression becomes:

$$\varphi (r) = \frac{Ae^{-qr} + Be^{qr}}{r} - \frac{C_2}{q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

The factor $F(R)$ is determined by equation 1. This equation becomes identical with equation 4, if we put $C_2 = 0$. The general solution becomes therefore:

$$F(R) = \frac{Me^{qR} + Ne^{-qR}}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

According to equation (2) the following equation holds good for function $F(R)$:

$$2 F'(R) + R F''(R) = C_2 \left(M e^{qR} + N e^{-qR} \right)$$

We find easily:

$$R F(R) = \frac{C_2}{q^2} \left(M e^{qR} + N e^{-qR} \right) + E R + D \quad . \quad . \quad . \quad (8)$$

in which E and D represent constants.

If in equation a we substitute the expressions we have found, for $\varphi(r)$, $F(R)$ and $F(R)$, we find the relations which must exist between the constants. We shall easily find:

$$M = \frac{1}{2q} \qquad N = -\frac{1}{2q}$$

Therefore

$$\varphi(r) = \frac{A e^{-qr} + B e^{qr}}{r} - \frac{C_2}{q^2} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$F(R) = \frac{e^{qR} - e^{-qR}}{2 qR} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The potential for a spherical shell in point P (see fig. 1), becomes therefore [1]:

$$M F(R) \varphi(r) = M \frac{e^{qR} - e^{-qR}}{2 qR} \times \frac{A e^{-qr} + B e^{qr}}{r} .$$

If ϱ is the density, then $4 \pi R^2 dR \varrho = M$. For the whole sphere we get therefore for the potential in an outside point:

$$4 \pi \varrho \, \varphi(r) \int_0^R F(R) R^2 dR = 4 \pi \varrho \frac{A e^{-qr} + B e^{qr}}{2 qr} \int_0^R R \left(e^{qR} - e^{-qR} \right) dR =$$

$$= 2 \pi \varrho \frac{A e^{-qr} + B e^{qr}}{qr} \left\{ \frac{1}{q} \left(R - \frac{1}{q} \right) e^{qR} + \frac{1}{q} \left(R + \frac{1}{q} \right) e^{-qR} \right\}$$

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[1] We put the constant of the potential function $= 0$.

or, if we substitute λ for $\dfrac{1}{q}$:

$$2 \pi \varrho \lambda^2 \frac{Ae^{-\frac{r}{\lambda}} + Be^{\frac{r}{\lambda}}}{r} \left\{ (R - \lambda) e^{\frac{R}{\lambda}} + (R + \lambda) e^{-\frac{R}{\lambda}} \right\} .$$

Starting from the function $\varphi(r) = - f \dfrac{e^{-\frac{r}{\lambda}}}{r}$: (Thermodynamische Theorie der Kapillar. Zeitschrift für phys. Chem. XIII, 4, 1894 p 721) Prof. VAN DER WAALS finds:

$$P = - 2 \pi f \varrho \lambda^2 \frac{e^{-\frac{r}{\lambda}}}{r} \left\{ (R - \lambda) e^{\frac{R}{\lambda}} + (R + \lambda) e^{-\frac{R}{\lambda}} \right\} .$$

The coefficient is the same as in the more general form of the potential function. If we take the more general expression

$$\varphi(r) = \frac{Ae^{-\frac{r}{\lambda}} + Be^{\frac{r}{\lambda}}}{r}$$ for $B = 0$ and $A = - f$, we get the function of VAN DER WAALS.

The theory of capillarity requires forces, which decrease with the distance and are attractive. The latter condition furnishes:

$$- \varphi'(r) \text{ negative or } \varphi'(r) \text{ positive.}$$

We have:

$$\varphi'(r) = - \frac{Ae^{-\frac{r}{\lambda}} + Be^{\frac{r}{\lambda}}}{r^2} + \frac{- Ae^{-\frac{r}{\lambda}} + Be^{\frac{r}{\lambda}}}{\lambda r}$$

So :

$$Ae^{-\frac{r}{\lambda}} \left(1 + \frac{r}{\lambda} \right) < Be^{\frac{r}{\lambda}} \left(-1 + \frac{r}{\lambda} \right)$$

for all positive values of r.

If we take $r = \lambda$, we get:

$$\frac{2A}{e} < Be \times 0$$

from which follows that A must be negative. Put $A = - f$, in which f represents a positive value, the last inequality but one becomes :

$$- fe^{-\frac{r}{\lambda}} \left(1 + \frac{r}{\lambda} \right) < Be^{\frac{r}{\lambda}} \left(\frac{r}{\lambda} - 1 \right)$$

or

$$-f \frac{\dfrac{1}{r}+\dfrac{1}{\lambda}}{\dfrac{1}{r}+\dfrac{1}{\lambda}+\dfrac{1}{1.2}\dfrac{r}{\lambda^2}+\dfrac{1}{1.2.3}\dfrac{r^2}{\lambda^3}+\cdots} < Be^{\frac{r}{\lambda}}\left(\frac{r}{\lambda}-1\right)$$

For always increasing values of r, the left hand member tends to zero and the coefficient of B becomes infinite. Therefore (symbolically):

$$-0 < B \times \infty$$

So B cannot be negative.
The former condition furnishes:

$$\frac{d\,\varphi'(r)}{dr} < 0$$

$$\frac{-2fe^{-\frac{r}{\lambda}}+2Be^{\frac{r}{\lambda}}}{r^3} - \frac{fe^{-\frac{r}{\lambda}}+Be^{\frac{r}{\lambda}}}{\lambda r^2} - \frac{fe^{-\frac{r}{\lambda}}+Be^{\frac{r}{\lambda}}}{\lambda r^2} + \frac{-fe^{-\frac{r}{\lambda}}+Be^{\frac{r}{\lambda}}}{\lambda^2 r} < 0$$

or

$$B\left\{\lambda^2+(\lambda-r)^2\right\}e^{\frac{r}{\lambda}} < f\left\{\lambda^2+(\lambda+r)^2\right\}e^{-\frac{r}{\lambda}}$$

(A is replaced by $-f$).
If r is always more increased, the left hand member becomes infinite, whereas the right hand member decreases infinitely.
Therefore symbolically

$$B \times +\infty < f \times +0.$$

So B cannot be positive.
As therefore B can be neither negative nor positive, B must be 0.
The function of VAN DER WAALS is the most general function which fulfils the conditions of the theory of capillarity and possesses the above mentioned property.
In answer to a letter on the subject discussed here, Prof. VAN DER WAALS and Prof. KORTEWEG were so kind to draw my attention to the work of Dr. C. NEUMANN: „Allgemeine Untersuchungen über das NEWTON'sche Princip der Fernwirkungen mit besonderer Rücksicht auf die elektrischen Wirkungen" Leipzig B. G. TEUBNER, 1896.

The principal problem, discussed by the author is as follows:
What must be the form of the potential function of electric agens,
spread over different conductors, in order to make an electrostatic
equilibrium possible. (The possibility of such an equilibrium is con-
sidered as an axiom).

Dr. NEUMANN finds as the most general potential function:

$$\varphi(r) = \frac{Ae^{-\alpha r}}{r} + \frac{Be^{-\beta r}}{r} + \frac{Ce^{-\gamma r}}{r} + \ldots$$

α, β, γ ... A, B and C are unknown positive and negative
quantities.

After this the conditions are inquired into, which these quantities
must fulfil, in order to make an equilibrium possible. The result
may be expressed in this thesis:

„Es sei gegeben irgend ein System von Conductoren und Isola-
toren. Jeder Isolator sei mit einer *festen* electrischen Vertheilung in
seinem Innern, und zugleich mit einer *festen* electrischen Belegung
an seiner Oberfläche versehen. Andererseits sei jeder Conductor
entweder zur Erde abgeleitet, oder aber isolirt und mit einer gege-
benen Elektricitätsmenge geladen.

Alsdann wird für dieses System, unter Zugrundelegung der Poten-
tialfunction:

$$\varphi(r) = \frac{Ae^{-\alpha r}}{r} + \frac{Be^{-\beta r}}{r} + \frac{Ce^{-\gamma r}}{r} + \ldots$$

stets ein elektrischer Gleichgewichtszustand existiren, falls nur die
Constanten α, β, γ... alle *positiv*, und die Constanten A, B, C...
alle von *einerlei* Vorzeichen sind.

Zu diesen Bedingungen wird offenbar für den besondern Fall,
dass die Reihe ins Unendliche fortschreitet, noch die hinzuzufügen
sein, dass die Reihe convergirt, sowie auch die, dass Integral

$$\int \varphi(r)\, \varepsilon\, d\tau$$

{ε = cubic density and $d\tau$ = element of volume}

einen bestimmten Sinn habe.

In that particular case that we restrict ourselves to the first
term, and so assume as potential the form $\varphi(r) = \dfrac{Ae^{-\alpha r}}{r}$ the

function is simply the potential function of VAN DER WAALS $\left(A = -f \text{ and } \alpha = \dfrac{1}{\lambda}\right)$.

For this case Dr. NEUMANN points out the property which Prof. VAN DER WAALS has also found for his potential function. (Zeitschrift für phys. Chemie XIII, 4, 1894, p. 721). He states his thesis as follows:

Die Einwirkung einer homogenen materiellen Kugelfläche auf *äussere* Punkte wird, bei Zugrundelegung des Gesetzes:

$$\varphi\,(r) = \frac{Ae^{-\alpha r}}{r}$$

genau dieselbe sein, als rührte sie her von einem einzigen im Centrum der Fläche befindlichen materiellen Punkt.

Und zwar hat die Masse M dieses der gegebenen Fläche äquivalenten materiellen Punktes den Werth:

$$M = M \frac{e^{\alpha R} - e^{-\alpha R}}{2\,\alpha\,R} = M\left(1 - \frac{(\alpha R)^2}{\pi\,3} + \frac{(\alpha R)^4}{\pi\,5} + \cdots\right)^{[1]}$$

wo M die Gesammtmasse der gegebenen Fläche, und R den Radius derselben bezeichnet. Es ist mithin:

$$M \geqq M$$

und zwar wird der Fall $M = M$ nur dann eintreten, wenn die Constante α des Gesetzes $\varphi\,(r) = \dfrac{Ae^{-\alpha r}}{r}$ verschwindet, jenes Gesetz also in das NEWTON'sche Gesetz sich verwandelt.

Moreover the remark is made, that if as potential the general function

$$\varphi\,(r) = \frac{A\,e^{-\alpha r}}{r} + \frac{B\,e^{-\beta r}}{r} + \frac{C\,e^{-\gamma r}}{r} +$$

is taken, also in this case a homogeneous spherical shell acts on an exterior point, as if the mass:

$$M = M\left\{\frac{e^{\alpha R} - e^{-\alpha R}}{2\,\alpha\,R} + \frac{e^{\beta R} - e^{-\beta R}}{2\,\beta\,R} + \cdots\right\}$$

were concentrated in the centre.

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[1] $\pi\,(n) = 1.\,2.\,3 \ldots n = $ Function of GAUSS.

I must, however, point out that this thesis is not of the same nature as that proved by Prof. van der Waals for the function $\varphi(r) = \dfrac{Ae^{-\alpha r}}{r}$. In this case namely we must be able to find the potential in the point in question by a simple multiplication of the potential function with a coëfficient, depending on the radius of the shell. This is not the case with the general function.

Every *term* of the function $\varphi(r)$ must be multiplied with the corresponding coefficient, to get the total potential. So it remains a superposition of different potentials.

The problem which I have treated: what must be the form of the potential function, by which a spherical shell acts on exterior points, as if (leaving a coefficient out of account), the mass was concentrated in the centre, was not discussed.

The second problem which I have tried to solve, is this. Is there a potential function which possesses the property just mentioned, while it is constant for a point inside the shell.

For the potential of a sperical shell at an exterior point we have found:

$$\frac{1}{2}\ \frac{M}{Rr}\int_{r-R}^{r+R} p\,\varphi(p)\,dp \qquad \varphi = \text{potential function.}$$

For an interior point we should have got:

$$\frac{1}{2}\ \frac{M}{Rr}\int_{R-r}^{R+r} p\,\varphi(p)\,dp\ .$$

If we put $\displaystyle\int r\,\varphi(r)\,dr = \psi(r)$, we get

$$V = \frac{M}{2\,Rr}\left\{\psi(R+r) - \psi(R-r)\right\}\ .$$

The general form for the potential function, fulfilling the first condition was:

$$\varphi(r) = \frac{A\,e^{-qr}}{r} + \frac{B\,e^{qr}}{r} + C\ .$$

A, B, C and q are arbitrary constants. The mass-coefficient depends

only on q. Now the question is: is it possible to choose those constants in such a way that the potential of a spherical shell with a radius R becomes constant for points inside the shell?

$$\psi\,(r) = \int^r r\,\varphi\,(r)\,dr = \int (A\,e^{-qr} + B\,e^{qr} + Cr)\,dr =$$

$$= -\frac{A}{q}\,e^{-qr} + \frac{B}{q}\,e^{qr} + \frac{1}{2}\,Cr^2.$$

So

$$V = \frac{M}{2\,Rr}\left\{ -\frac{A}{q}\,e^{-q(R+r)} + \frac{B}{q}\,e^{q(R+r)} + \frac{1}{2}\,C(R+r)^2 + \right.$$

$$\left. + \frac{A}{q}\,e^{-q(R-r)} - \frac{B}{q}\,e^{-q(R-r)} - \frac{1}{2}C(R-r)^2 \right\}.$$

The expression must now depend on r. It is easy to see that we have only to take $A = f\,e^{qR}$ and $B = -f\,e^{-qR}$, to get ($f =$ constant):

$$V = \frac{M}{2\,Rr}\left\{ -\frac{f}{q}\,e^{-qr} - \frac{f}{q}\,e^{qr} + \frac{f}{q}\,e^{qr} + \frac{f}{q}\,e^{-qr} + 2\,CRr \right\} = M\,.\,C$$

in which we have also fulfilled the second condition.

The potential function becomes therefore:

$$\varphi\,(r) = f\,e^{qR}\frac{e^{-qr}}{r} - f\,e^{-qR}\frac{e^{qr}}{r} + C\,.$$

Considered superficially we now get in contradiction with the theorem of LAPLACE, which states that the law of NEWTON is the only law which fulfils the condition, that the spherical shell exercises no force on a point inside it. In reality this theorem includes more. The function of forces must namely keep this property without change of the constant, whatever the radius of a spherical shell may be. However in the case discussed by us the radius of the shell is given and in the potential we have therefore introduced constants depending on the radius of the shell.

As solution of equation (4) we found two integrals. If we had substituted function (6a) in equation a and if, in the same way as before, we had sought the conditions which the different coefficients must fulfil, we should have found that $C = 0$ and further

$$F\,(R) = \frac{\sin q\,R}{q\,R}\,.$$

The potential of a spherical shell in point P becomes therefore:

$$M \frac{\sin q R}{q R} \frac{A_1 \sin (qr + \alpha)}{r} \,^1) .$$

Though the function $\dfrac{A_1 \sin (qr + \alpha)}{r}$ is of no importance for the theory of the molecular forces, it has nevertheless another remarkable physical signification.

By twice differentiating with respect to x, we shall easily find :

$$\frac{d^2 \varphi}{dx^2} = - \frac{A_1 \sin(qr + \alpha)}{r^3} + \frac{3 A_1 \sin (qr + \alpha) x^2}{r^5} - \frac{3 A_1 q \cos (qr + \alpha) x^2}{r^4} +$$

$$+ \frac{A_1 q \cos (qr + \alpha)}{r^2} - \frac{A_1 q^2 \sin (qr + \alpha) x^2}{r^3} .$$

In the same way we find for $\dfrac{d^2 \varphi}{dy^2}$ and $\dfrac{d^2 \varphi}{dz^2}$ corresponding expressions. By adding these equations, we get:

$$\nabla^2 \varphi = - \frac{A_1 q^2 \sin (qr + \alpha)}{r} = - q^2 \varphi . \quad . \quad . \quad . \quad (11)$$

As is well known this differential equation is of great importance in the theory of the conduction of heat. The function found is an extension of the caloric potential of MATHIEU.

If we had deduced for the first found function $\varphi (r) = \dfrac{A e^{-qr} + B e^{qr}}{r}$ the second differential coefficient according to x, we should have found :

$$\frac{d^2 \varphi}{dx^2} = - \frac{A e^{-qr} + B e^{qr}}{r^3} + \frac{3 q (A e^{-qr} - B e^{qr}) x^2}{r^4} + \frac{3 (A e^{-qr} + B e^{qr}) x^2}{r^5} -$$

$$- \frac{q (A e^{-qr} - B e^{qr})}{r^2} + \frac{q^2 (A e^{-qr} + B e^{qr}) x^2}{r^3} .$$

If we calculate in the same way the corresponding expressions for $\dfrac{d^2 \varphi}{dy^2}$ and $\dfrac{d^2 \varphi}{dz^2}$, we find by putting the three quantities together:

$^1)$ In putting $q = 0$ we refind the potential which we should have found according to the law of attraction of NEWTON.

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$$\nabla^2 \varphi = q^2 \frac{A\,e^{-qr} + B\,e^{qr}}{r} = q^2 \varphi \quad . \quad . \quad . \quad . \quad (12)$$

In the particular case that $A = -\dfrac{1}{2q}$ and $B = \dfrac{1}{2q}$, the relation between the two equations (11) and (12) is evident. The function $\varphi(r) = \dfrac{A\,e^{-qr} + B\,e^{qr}}{r}$ becomes: $\dfrac{e^{qr} - e^{-qr}}{2\,qr}$. If we substitute $q\sqrt{-1}$ for q, the latter expression becomes:

$$\frac{e^{q r\sqrt{-1}} - e^{-qr\sqrt{-1}}}{2\,qr\,\sqrt{-1}} = \frac{\sin qr}{qr} \,.$$

This function is a special case of the more general

$$\varphi(r) = \frac{A_1 \sin(qr + \alpha)}{r} \,. \quad A_1 = \frac{1}{q} \quad \text{and} \quad \alpha = 0 \,.$$

By substituting in equation (12) $q\sqrt{-1}$ for q, we get equation (11). If $q = 0$ the two equations yield the well known equation:

$$\nabla^2 = 0 \,.$$

The functions

$$\varphi(r) = \frac{A\,e^{-qr} + B\,e^{qr}}{r} \quad \text{and} \quad \varphi(r) = \frac{A_1 \sin(qr + \alpha)}{r}$$

are solutions of two *different* partial differential equations of the 2nd order, but we have seen that they are also common solutions of the *same* problem.

We might also have deduced the partial differential equation (12) in the following way:

$$\frac{A\,e^{-ar}}{r} = \frac{A}{r}\left(1 - qr + \frac{q^2 r^2}{\pi 2} - \frac{q^3 r^3}{\pi 3} + \cdot\cdot\right) =$$

$$= \frac{A}{r} - Aq + \frac{A\,q^2 r}{\pi 2} - \frac{A\,q^3 r^2}{\pi 3} + \cdot\cdot$$

and

$$\frac{B\,e^{qr}}{r} = \frac{B}{r}\left(1 + qr - \frac{q^2 r^3}{\pi 2} + \frac{q^3 r^3}{\pi 3} + \cdot\cdot\right) =$$

$$= \frac{B}{r} + Bq + \frac{B\,q^2 r}{\pi 2} + \frac{B\,q^3 r^2}{\pi 3} + \cdot\cdot$$

So :

$$\frac{Ae^{-qr} + Be^{qr}}{r} - \frac{A + B}{r} = q(B - A) + \frac{(A + B)q^2 r}{\pi\,2} + \frac{(B-A)\,q^3\,r^2}{\pi\,3} + \cdots$$

If we apply to the two members of this equation the operation ∇^2, we find

$$\nabla^2 \left\{ \frac{Ae^{-qr} + Be^{qr}}{r} = q^2 \left\{ \frac{A + B}{r} + (B - A)\,q + \frac{(A + B)\,q^2\,r}{\pi\,2} + \right\} = \right.$$

$$= q^2 \frac{Ae^{-qr} + Be^{qr}}{r}$$

In the same way equation (11) may also be deduced.

On the other hand, if $x^2 + y^2 + z^2 = r^2$, it is possible to show that the solutions of the differential equations $\nabla^2 \varphi = \pm\, q^2\, \varphi$, give exactly those functions that possess the property found by Prof. VAN DER WAALS for the potential function $-f \dfrac{e^{-qr}}{r}$. We have, viz:

$$\frac{d\varphi}{dx} = \frac{d\varphi}{dr}\,\frac{dr}{dx} \qquad \frac{d^2\varphi}{dx^2} = \frac{d^2\varphi}{dr^2} \left(\frac{dr}{dx}\right)^2 + \frac{d\varphi}{dr}\,\frac{d^2r}{dx^2}$$

and because $r^2 = x^2 + y^2 + z^2$:

$$\frac{dr}{dx} = \frac{r}{r} \qquad \frac{d^2r}{dx^2} = \frac{1}{r} - \frac{x^2}{r^3}$$

therefore

$$\frac{d^2\varphi}{dx^2} = \frac{d^2\varphi}{dr^2}\,\frac{x^2}{r^2} + \frac{d\varphi}{dr} \left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

If we deduce the expressions for $\dfrac{d^2\varphi}{dy^2}$ and $\dfrac{d^2\varphi}{dz^2}$ in the same way, we find by addition:

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\,\frac{d\varphi}{dr} = 0\ .$$

The differential equations $\nabla^2 \varphi = \pm\, q^2\, \varphi$ may therefore be written in the following way:

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\,\frac{d\varphi}{dr} = \pm\, q^2\, \varphi \quad \cdots \cdots \quad (13)$$

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We get these equations, when for C_1 we substitute $+ q^2$ and $- q^2$ respectively in equation (4) and when we put $C_2 = 0$, and this proves that the solutions of (13) fulfil the condition in question.

In a further paper I hope to prove the two following theorems:

I. If in a region of space ϱ and v are functions of x, y and z, and v satisfies the three following conditions:

1st v and its differential coefficients with respect to x, y and z are everywhere continuous;

2nd with the exception of some points or surfaces in this space

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = q^2 v - 4 \pi (A + B) \varrho \; ;$$

3rd the products xv, yv, zv, $x^2 \dfrac{dv}{dx}$, $y^2 \dfrac{dv}{dx}$ en $z^2 \dfrac{dv}{dz}$ are nowhere infinite;

then v is the potential with respect to the point x, y and z of an agens, the density of which is ϱ, while the potential function is expressed by:

$$\varphi (r) = \frac{A e^{-qr} + B e^{qr}}{r} \; .$$

II. If the same conditions as in I hold for ϱ and v with this modification that $- q^2$ is substituted for q^2 and $A \sin \alpha$ for $A + B$;

then v is the potential with respect to point x, y and z of an agens, the density of which is ϱ, while the potential function is expressed by

$$\varphi (r) = \frac{A \sin (qr + \alpha)}{r} \; .$$

Hydrography. — *Tidal Constants in the Lampong- and Sabang-bay, Sumatra.* By Dr. J. P. VAN DER STOK.

I. *Telok Betong.*

a. From April 23,1897 to April 22,1898 tidal observations have been made in the Lampong-bay on the road of *Telok Betong*, situated in 5° 27' Lat. S. and 105° 16' Long. E. at the 6 hours of 8 and 10 a. m., noon 2, 4 and 6 p. m.

As in the eastern parts of Sunda-strait the normal (i. e. oceanic) tides of the Indian Ocean must show a more or less gradual trans-