Citation:

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Partly neglecting the variations of the monthly tide-ranges in different years, which are at a maximum 120 mm. and give only a maximum error of 3 mm. in the value of A, we can put $^2/_{75} \times 2750 + a_2 \cos(\varphi - \gamma_2)$ instead of $^2/_{75}\ V - {}^2/_{75}\ E$ mm.

The formula then becomes:

$$A = 78 - 0{,}08\,E + a_3 \cos(\varphi - \gamma_3) + Y.$$

The errors calculated from this formula do not differ much from the above-mentioned, for now the mean error is 6,4 mm. and the formula is therefore as exact, while the computation is much more easily carried out.

In conclusion I will add the following remarks.

First I should mention that I found some errors in the tables containing the observed height of the sea-level at Delfzyl during the months in which the greatest differences occurred and in those of two other tide-gauges during five months. For instance at Delfzyl I found a month in which one height had been read from the half hour-mark instead of the hour-mark, and also one reading with a wrong sign. After making the correction the great divergence was very much reduced. Although this is no proof, we may suppose that the greatest differences very nearly give the limit of precision.

Further I notice that the second correction mentioned above does not agree with the principle on which the method of harmonic analysis is founded, so that this method cannot give exact results in the reduction of the observations at Delfzyl. Still, I do not affirm that any other is better.

This want of agreement is demonstrated by the term 0,08 E in the preceding formula. For the same month in two different years (February 1889 and 1890) the difference of the two values of E is 583 mm., so that 0,08 E = 47 mm., and although this difference would not be of much importance for a single observation, it is far too great for an error of the monthly mean.

Mathematics. — Prof. JAN DE VRIES reads for Prof. L. GEGENBAUER at Vienna a paper entitled: "*New theorems on the roots of the functions* $C_n^{\nu}(x)$".

Up to this moment we know of the roots of the coefficients $C_n^{\nu}(x)$ of the development of $(1 - 2\alpha + \alpha^2)^{-\nu}$ according to ascending powers of α only this, that they are all real and unequal, are situated between $+1$ and -1 and — apart from the root 0 appearing in the case of an uneven n — have in pairs the same absolute value; finally that the roots of $C_n^{\nu}(x)$ and $C_{n-1}^{\nu}(x)$ as well as those of $C_n^{\nu}(x)$ $C_{n-1}^{\nu+1}(x)$ mutually separate each other.

In the following lines some new theorems on the roots of these functions will be found in a highly simple manner, one of which including as a special case a well known theorem of the theory of spherical functions.

1. From the addition-theorem of the functions $C_n^{\nu}(x)$ arrived at by me

$$C_n^{\nu}[xx_1+\sqrt{(1-x^2)}\sqrt{(1-x_1^2)}\cos\varphi]=\Pi(2\nu-2)\left[\frac{2^n\,\Pi(n+\nu-1)}{\Pi(\nu-1)}\right]^2$$

$$\sum_{\rho=0}^{\rho=n}(-1)\frac{2\nu+2\rho-1}{\Pi(n-\rho)\,\Pi(n+2\nu+\rho-1)}\,C_n^{\nu,\rho}(x)\;C_n^{\nu,\rho}(x_1)\;C_\rho^{\frac{2\nu-1}{2}}(\cos\varphi)\;,$$

$$(1\geqq x,\;x_1\geqq-1)$$

where the square roots are taken positively and

$$C_n^{\nu,\rho}(x)=\frac{\Pi(n-\rho)\,\Pi(\nu+\rho-1)}{2^{n-\rho}\,\Pi(n+\nu-1)}(x^2-1)^{\frac{\rho}{2}}\,C_{n-\rho}^{\nu+\rho}(x)\;,$$

we find the relation

$$\int_0^{\pi}C_n^{\nu}[xx_1+\sqrt{(1-x^2)}\sqrt{(1-x_1^2)}\cos\varphi]\sin^{2\nu-1}\varphi\,d\varphi=$$

$$=\frac{2^{2\nu-1}[\Pi(\nu-1)]^2\,\Pi(n)}{\Pi(n+2\nu-1)}\;C_n^{\nu}(x)\;C_n^{\nu}(x_1)\,.$$

By putting x_1 equal to a positive root z_n of the function $C_n^{\nu}(x)$ the equation is transformed into

$$\int_0^{\pi}C_n^{\nu}[xz_n+\sqrt{(1-x^2)}\sqrt{(1-z_n^2)}\cos\varphi]\sin^{2\nu-1}\varphi\,d\varphi=0\;,$$

showing that the function $C_n^{\nu}(x)$ vanishes at least for one value of its argument lying between $xz_n+\sqrt{(1-x^2)}\sqrt{(1-z_n^2)}$ and $xz_n-\sqrt{(1-x^2)}\sqrt{(1-z_n^2)}$, as otherwise the function to be integrated would not change its sign in the entire region of integration, and hence the integral could not be equal to 0. This value certainly differs from z_n when

$$z_n > x z_n + \sqrt{(1-x^2)}\sqrt{(1-z_n^2)},$$

which can only be the case, if

$$x < 2 z_n^2 - 1,$$

and this leads, in case x might also be positive, to the supposition

$$z_n > \frac{1}{\sqrt{2}}.$$

The entire inteival under discussion being a positive one when x is taken greater than $\sqrt{(1-z_n^2)}$, we find the theorem:

If z_n be a positive root of the function $\overset{\nu}{\underset{n}{C}}(x)$ surpassing $1:\sqrt{2}$ and α a positive root lying between $|\sqrt{(1-z_n)}|$ and $2z_n^2-1$, then there must be in the interval $\alpha z_n - \sqrt{(1-\alpha^2)}\sqrt{(1-z_n^2)}$ to $\alpha z_n + \sqrt{(1-\alpha^2)}\sqrt{(1-z_n^2)}$ at least one other positive root of this function (smaller than z_n).

A corollary of this theorem is the following:

The smallest positive root of the function $\overset{\nu}{\underset{n}{C}}(x)$ is smaller than $1:\sqrt{2}$.

2. In my paper "Some theorems on the functions $\overset{\nu}{\underset{n}{C}}(x)$" ("Einige Sätze über die Functionen $\overset{\nu}{\underset{n}{C}}(x)$") contained in the 47th vol. of "Denkschriften der mathematisch-naturwissenschaftlichen Classe der Kais. Akademie der Wissenschaften in Wien" I have given the four following equations:

$$\overset{\nu}{\underset{n}{C}}(\cos x) = \frac{(-1)^n \Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\sin^{2\nu-1}\frac{x}{2}} \int_0^x (\cos\varphi - \cos\vartheta)^{\nu-1}$$

$$\overset{2\nu}{\underset{2n}{C}}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi \quad (\nu > 0),$$

$$\overset{\nu}{C}_n(\cos x) = \frac{\Pi\left(\frac{2\nu-1}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-1)\cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi} (\cos x - \cos\varphi)^{\nu-1} \overset{2\nu}{C}_{2n}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi \quad (\nu > 0),$$

$$\overset{\nu}{C}_n(\cos x) = \frac{(-1)^n\,\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\sin^{2\nu-1}\frac{x}{2}} \int_0^{x} (\cos\varphi - \cos x)^{\nu-1} \overset{2\nu-1}{C}_{2n+1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi \quad (\nu \geqq \tfrac{1}{2}),$$

$$\overset{\nu}{C}_n(\cos x) = \frac{\Pi\left(\frac{2\nu-3}{2}\right)}{2^{\nu-1}\sqrt{\pi}\,\Pi(\nu-2)\cos^{2\nu-1}\frac{x}{2}} \int_x^{\pi} (\cos x - \cos\varphi)^{\nu-1} \overset{2\nu-1}{C}_{2n+1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi \quad (\nu \geqq \tfrac{1}{2}),$$

which are a generalization of the integrals:

$$P_n(\cos x) = \frac{2}{\pi}\int_0^x \frac{\cos(n+\frac{1}{2})\varphi\,d\varphi}{\sqrt{2(\cos\varphi - \cos x)}},$$

$$P_n(\cos x) = \frac{2}{\pi}\int_x^\pi \frac{\sin(n+\frac{1}{2})\varphi\,d\varphi}{\sqrt{2(\cos x - \cos\varphi)}}.$$

given by MEHLER in his communication "Notice on integralforms of Dirichlet for the spherical functions $P_n(\cos\vartheta)$ and an analogous integralform for the cylindrical functions $I(x)$", "(Notiz über die Dirichlet'schen Integralausdrücke für die Kugelfunctionen $P_n(\cos\vartheta)$ und eine analoge Integralform für die Cylinderfunctionen $I(x)$)". By putting x equal to the root of the function $\overset{\nu}{C}_n(\cos x)$ lying between 0 and $\frac{\pi}{2}$ we transform them into the following relations:

$$\int^{y_n}(\cos\varphi-\cos y_n)^{\nu-1}\,C_{2n}^{2\nu}\left(\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,d\varphi=0\ ,$$

$$\int_{y_n}^{\pi}(\cos y_n-\cos\varphi)^{\nu-1}\,C_{2n}^{2\nu}\left(\cos\frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\,d\varphi=0\ ,$$

$$\int_0^{y_n}(\cos\varphi-\cos y_n)^{\nu-1}\,C_{2n+1}^{2\nu-1}\left(\sin\frac{\varphi}{2}\right)\sin\varphi\,d\varphi=0\ ,$$

$$\int_{y_n}^{\pi}(\cos y_n-\cos\varphi)^{\nu-1}\,C_{2n+1}^{2\nu-1}\left(\cos\frac{\varphi}{2}\right)\sin\varphi\,d\varphi=0\ ,$$

which relations show that the functions $C_{\varkappa}^{\mu}\left(\sin\frac{\varphi}{2}\right)$ and $C_{\varkappa}^{\mu}\left(\cos\frac{\varphi}{2}\right)$ vanish at least for one value of φ within the respective interval of integration. This gives rise to the following theorems:

The smallest among the roots of $C_n^{\nu}(\cos x)$ *lying between* 0 *and* $\frac{\pi}{2}$ *is larger than the smallest of the roots of* $C_{2n}^{2\nu}\left(\cos\frac{x}{2}\right)$ *fulfilling the same conditions and the greatest among the above named roots of* $C_n^{\nu}(\cos x)$ *is smaller than the greatest among the roots of* $C_{2n}^{2\nu}\left(\cos\frac{x}{2}\right)$ *belonging to this region.*

The smallest among the roots of $C_n^{\nu}(\cos x)$ *lying between* 0 *and* $\frac{\pi}{2}$ *is larger than the smallest of the roots of* $C_{2n+1}^{2\nu-1}\left(\sin\frac{x}{2}\right)$ *fulfilling the same conditions, and the greatest among the above named roots of* $C_n^{\nu}(\cos x)$ *is smaller than the greatest among the roots of* $C_{2n+1}^{2\nu-1}\left(\cos\frac{x}{2}\right)$ *belonging to this region:*

By putting in the first proposition ν equal to ½ and by marking that

$$\overset{\frac{1}{2}}{\underset{n}{C}}(\cos x) = P_n(\cos x) \ ,$$

$$\overset{1}{\underset{2n}{C}}(\cos \chi) = \frac{\sin (2n+1)\chi}{\sin \chi},$$

$$\overset{1}{\underset{2n}{C}}(\sin \chi) = (-1)^n \frac{\cos (2n+1)\chi}{\cos \chi}$$

and that sin α increases, $\cos \alpha$ however diminishes with α, we arrive at the theorem:

The positive roots of the n^{th} spherical function $P_n(x)$ lie between $\cos \frac{n_1 \pi}{2n+1}$ and $\cos \frac{\pi}{2n+1}$ where n_1 is the greatest even number contained in n.

This theorem is a corollary of the one deduced by Bruns in his treatise "On the theory of the spherical functions" (Zur Theorie der Kugelfunctionen) published in the 90^{th} vol. of Crelle's Journal and recently proved by Markoff [1]) and Stieltjes [2]):

The roots of the spherical functions lie one by one in the intervals
$\cos \frac{2i\pi}{2n+1} \ldots \cos \frac{(2i-1)\pi}{2n+1}$.

From the preceding theorems we can easily deduce the following:

The difference between the greatest positive root of the function $\overset{\nu}{\underset{n}{C}}(x)$ and unity is less than two times the square of the smallest positive root of $\overset{2\nu}{\underset{2n}{C}}(x)$.

The difference between the greatest positive root of the function $\overset{\nu}{\underset{n}{C}}(x)$ and unity is less than two times the square of the smallest positive root of $\overset{2\nu-1}{\underset{2n+1}{C}}(x)$.

The difference between the greatest positive root of the n^{th} spherical function and unity is less than $2 \cos^2 \frac{n_1 \pi}{2n+1}$.

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[1]) "On the roots of certain equations", ("Sur les racines de certaines équations") Mathem. Annalen, 27th Vol.

[2]) "On the roots of the equation $X_n = 0$," ("Sur les racines de l'équation $X_n = 0$") Acta Mathematica, IX Vol. "On the polynomia of Legendre", ("Sur les polynomes de Legendre"), Annales de la Faculté des Sciences de Toulouse. Vol. IV. Markoff and Stieltjes deduce in the cited treatise also the narrower limits $\cos \frac{i\pi}{n+1} \ldots \cos \frac{(2i-1)\pi}{2n}$.

Two times the square of the smallest positive root of $C^{2\nu}_{2n}(x)$ *is smaller than the smallest positive root of* $C^{\nu}_{n}(x)$ *increased by* 1.

Two times the square of the smallest positive root of $C^{2\nu-1}_{2n+1}(x)$ *is smaller than the smallest positive root of* $C^{\nu}_{n}(x)$ *increased by* 1.

The two latter theorems furnish us with a less narrow limitation for the smallest positive root of the function $C^{\nu}_{n}(x)$ than the theorem at the conclusion of § 1.

Terrestrial magnetism. — Dr. W. VAN BEMMELEN. "*"Spasms" in the terrestrial magnetic force at Batavia.*" (Communicated by Prof. H. KAMERLINGH ONNES).

(Read September 30th 1899).

Since the great development of Seismology, the instruments, which record photographically the quantities determining the earth's magnetism have also rendered good service as Seismographs in the researches on the propagation of earth-waves in the surface of the earth.

During half a year I had the opportunity of tracing the seismic disturbances in the Magnetograms at Batavia, and this under very favourable circumstances; for, not only was the fear of local disturbance very small, the temperature constant and the damping large, but since June 1st 1898 a new Milne-Seismograph had been working and furnishing accurate information about seismic disturbances. When an earthquake is near, these appear in the curves of the Magnetograms as discontinuities, viz. the needle suddenly starts vibrating and continues doing so for some minutes; when at a greater distance, on the contrary, only a more or less considerable regular broadening of the curves appears. Comparison with the Milne-Seismograms quickly taught me that the seismic disturbances at Batavia seldom are large enough to appear in the Magnetograms, but also conversely, that no trace of a large number of analogous disturbances in the Magnetograms could be detected in the Seismograms.

Hence there is danger of considerable confusion: if for instance an earth-wave has passed at Batavia at 11.10 which has not appeared in the Magnetograms, then very likely a non-seismic disturbance, occurring at 11.5 for instance, will be mistaken for an earth-wave and an error of minutes will be made. Moreover it is necessary to inquire whether a new phenomenon does not mingle with those just mentioned.