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For the sake of comparison we may note, that formerly at the temperature 15° C. and in a field of 7700 C. G. S. units the increase was found to be:

in the position I 6,5 II 14,9

Physiology. — "*The resorption of fat and soap in the large and the small intestine*". By Dr. H. J. HAMBURGER.

(Will be published in the Proceedings of the next meeting).

Mathematics. — "*On an application of the involutions of higher order*". By Prof. J. CARDINAAL.

1. One of the best known problems of the theory of the pencil of conics is the determination of the number of particular conics in such a pencil, where one rectangular hyperbola, two parabolae and three pairs of straight lines are obtained. The corresponding problem of geometry in space, namely the determination of the number of particular quadric surfaces in a pencil of those quadrics (pencil of S^2), offers more difficulties.

It is true, it is easy to prove that there are three paraboloids in a pencil of S^2; but more difficult is it to trace the number of other particular groups of surfaces. The surfaces of revolution cannot be reckoned amongst these, having to satisfy two conditions. However, the orthogonal (rectangular) hyperboloids can be, as it will be proved that these are bound by one condition only.

My purpose in this communication is to investigate first how many rectangular hyperboloids appear in a general pencil of S^2 and consecutively to prove that the construction may be brought back to a problem of synthetic geometry in the plane, a problem where the theory of involutions of higher order must be applied.

2. According to definition an hyperboloid is rectangular when the cyclic planes are normal to two generatrices. With CLEBSCH [1]) we however think it preferable to choose a definition, in which we make use of the section of the hyperboloid with the plane at infinity. To investigate the rectangularity we set to work as follows:

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[1]) CLEBSCH-LINDEMANN: Vorlesungen uber Geometrie, ("Lessons on Geometry"), Vol. II, Part 1, p. 195, where we also find the literature of this subject mentioned.

first we determine the section (H^2) of the hyperboloid with the plane at infinity, then we construct the chords of intersection of H^2 with the imaginary circle (C^2) in that plane. If the pole of one of those chords of intersection in reference to C^2 falls in H^2, the hyperboloid is rectangular.

3. By this method the problem of space is transformed into a problem of the plane; in the further treatment, however, we come across the difficulty of an imaginary conic C^2. For a better insight into the problem, we substitute for the present an arbitrary plane for the plane at infinity, a real conic for the imaginary circle and then the problem is formulated as follows:

Given a conic K^2 and a pencil of conics with the also real base points 1, 2, 3, 4; to determine a conic L^2 of the pencil, for which the pole of a chord of intersection with K^2 lies on L^2.

4. If L^2 is found, we can still make the following remark about the solution: Let L^2 intersect the conic K^2 in the points L_1, L_2, L_3, L_4; let L_{12} be the pole of $L_1 L_2$ in reference to K^2 and let L^2 be brought through L_{12}, then according to a known theorem L^2 will also pass through the pole L_{34} of the opposite chord $L_3 L_4$ [1]). So the points 1, 2, 3, 4, L_{12}, L_{34} lie on the same conic. After this remark we can pass to the construction of the locus of the poles, supposing that L^2 describes the whole pencil.

5. Let A^2 be a conic of the pencil (1234): it intersects K^2 in the four points A_1, A_2, A_3, A_4. These four points will give rise to six common chords $A_1 A_2$, $A_1 A_3$, $A_1 A_4$, $A_2 A_3$, $A_2 A_4$, $A_3 A_4$ which correspond to six poles $A_{12} \ldots A_{34}$. If A^2 is replaced successively by all the conics of the pencil, every new conic gives rise to four new points of intersection: on K^2 these quadruples form an involution of the fourth order. It is clear, that if A_1 is chosen arbitrarily and conic A^2 is constructed, A_2, A_3, A_4 on K^2 are also determined, and that reciprocally when one of the last points, take A_2, is chosen, A_1, A_3 and A_4 are also determined. The six lines joining the quadruples of points by two envelop a curve C_3 of the third class [2]).

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1) STEINER-SCHRÖTER: Theorie der Kegelschnitte, ("Theory of Conics"), II, 3rd edition, p. 526, problem 90.

2) R. STURM, Die Gebilde ersten und zweiten Grades der Liniengeometrie, ("The figures of the first and the second order in the geometry of the straight line)", I, p. 29.

MILINOWSKI, Zur Theorie der kubischen und biquadratischen Involutionen, ("Theory of cubic and biquadratic involutions") Zeitschrift f. Math. und Physik, 19, p. 212 etc.

We can determine the order of this curve C_3 in the following way: Construct one of the common tangents t_1 of K^2 and C_3; let T_1 be the point of contact of t_1 with K^2, then T_1 is a double point of the involution. Through T_1 still two tangents can be drawn to C_3; they intersect K^2 in the branchpoints of the involution; these branchpoints are common points of K^2 and C_3. We can conclude from the number 6 of the common tangents that there are 12 of these branchpoints; therefore C_3 intersects the conic K^2 in 12 points. Hence C^3 is of the sixth order and may be called C^6.

6. The locus of the poles of the tangents of C^6 in reference to K^2 is the reciprocal polar curve C^3 of C^6; it is of the third order and of the sixth class. We imagine once more a point A_{12} on C^3 as the pole of chord $A_1 A_2$ of K^2; A_1 and A_2 determine two points of the corresponding conic A^2 of the pencil which intersects K^2 moreover in A_3 and A_4; A^2 also intersects C^3 in the six points $A'_{12}, A'_{13} \ldots A'_{34}$. Moreover five other poles will appear on C^3 besides A_{12}, the poles of the chords A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4. By assuming one point on C^3, two groups, each of 6, are determined on C^3, the group A and the group A'. If one of the points A is taken arbitrarily no point of group A will coincide with a point of group A'.

7. The following conclusions may be arrived at from the preceding:

a. If we assume successively the points A_{12}, B_{12}, $C_{12} \ldots$ on C^3, as many groups of 6 points are formed; each of the points of the group can determine the whole group unequivocally, so the points A, B, $C \ldots$ form an involution of the sixth order on C^3.

b. The points of intersection A', B', $C' \ldots$ of the conics with C^3 also form an involution of the sixth order.

c. Each point of group A' corresponds to any point of group A; reciprocally each point of group A corresponds to any point of group A'; so both involutions are projective.

d. The points of coincidence of both involutions determine the conics which give the solution of the problem (3).

8. The projective involutions on the same bearer are both of the sixth order, so they have 12 points of coincidence [1]. These points may be indicated more closely in the following manner:

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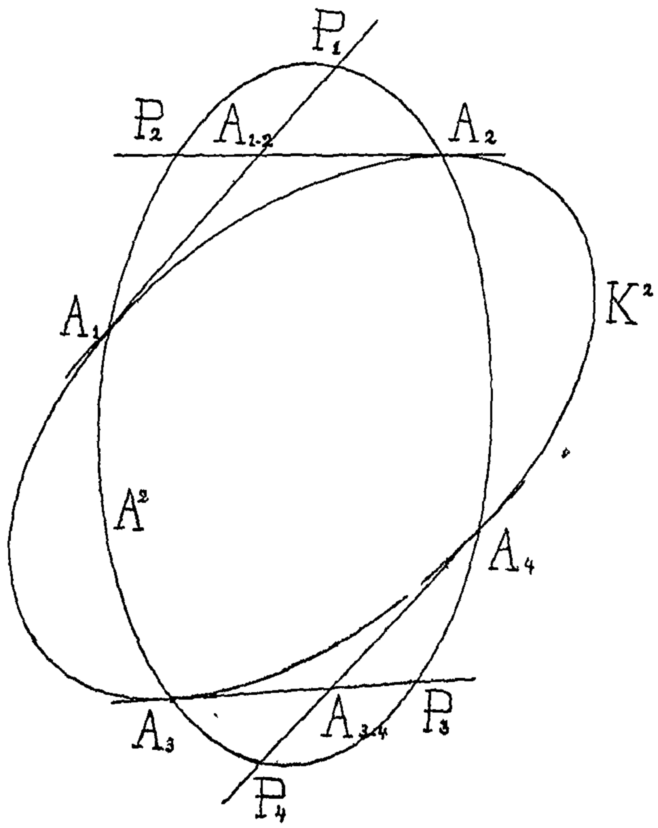
[1]) E. Kötter, Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven, ("Elements of a purely geometrical theory of the algebraic plane curves".) p. 88 etc.

C^3 intersects K^2 in 6 points; these 6 points are at the same time the points of contact on K^2 of the common tangents of K^2 and C^6. From this we can conclude that for these points the end-points of the chords of intersection coincide; so in these points conics of pencil (1 2 3 4) will touch K^2. If we imagine one of these points T_1 to be given, a conic of the pencil passes through this point, and as it touches K^2, the corresponding pole falls in T_1; from this we conclude that:

The 6 points of intersection $T_1 \ldots T_6$ of C^3 and K^2 are 6 of the points of coincidence of both involutions; so we have still to account for 6 other points of coincidence. Let us call one of these points A_{12}, then the conic through A_{12} will meet K^2 in A_1, A_2, A_3, A_4 and will pass through A_{34}, the pole of A_3 A_4. Hence these six points can be divided into three pairs of points $(A_{12}\ A_{34})$ $(B_{12}\ B_{34})$ $(C_{12}\ C_{34})$.

By the way we remark that the obtained result is in accordance with the fact, that six conics of a pencil touch an arbitrary conic.

9. The found three pairs of points determine the three conics which will solve the problem. We however add a second deduction,



which is connected, as will be proved, with the theory of double points of curves of higher order.

Let A^2 again be a conic of the pencil (1 2 3 4); we imagine two of the points of intersection of A^2 with K^2 to be constructed, take A_1 and A_2, and moreover the pole A_{12} of A_1 A_2 in reference to K^2 (see diagram).

The tangents A_1 A_{12} and A_2 A_{12} intersect the conic A^2 for the second time in the points P_1 and P_2; in the same way we can also determine the tangents in the points A_3 and A_4 with their second points of intersection P_3 and P_4. If A^2 describes the whole pencil, P_1, P_2, P_3, P_4 generate a locus; at the same time the poles A_{12} . . . A_{34} generate the locus C^3 found formerly. The conics forming the solution of the problem proposed sub (3) must now be brought through the points of intersection of the curve C^3 with the locus of the points P_1, P_2, P_3, P_4.

10. To determine the order of the locus lastly named, let us take a straight line l and determine how many points it has in common with it. We take a point A_1 on l, draw from that point two tangents to K^2 and construct the conic (1 2 3 4) through each of the points of contact; as we can construct two conics, four points of intersection A'_1, A'_2, A'_3, A'_4 on l will be found. So to one point A_1 belong 4 points A' in the just found correspondence.

Reversely if we construct the conic passing through A'_1, then it also passes through one of the other points A' say A'_2; it determines 4 points of intersection with K^2; the tangents to K^2 through those points determine still three points A_2, A_3, A_4 besides A_1. Consequently 4 points A' correspond to one point A and 4 points A to one point A'.

So there exists a projective correspondence (4,4) between these points A and A' which possesses 8 points of coincidence. So the required locus intersects l in 8 points, hence it is a curve of the 8^{th} order.

11. However, this curve breaks up into two parts. It is clear that K^2 itself belongs to the locus of the points of intersection of the tangents to K^2 with the variable conic A^2. The remaining curve will be of the 6^{th} order; we have now to investigate its particular points. These are the following:

a. The points 1, 2, 3, 4 are double points of K^6. To prove this we consider point 1, from which we draw the tangents t_1 and t_2 to K^2. There is a conic of the pencil (1 2 3 4) passing through the point of contact t_1 and a second through the point of contact t_2; if the variable conic describes the pencil, then the locus will pass two times through the point 1; so 1 is a double point and so are 2, 3 and 4.

b. The points of intersection of K^6 and C^3 are double points. C^3 is the locus of the poles $A_{12} \ldots A_{34}$; in these poles two tangents concur; so the locus also passes two times through these poles. It is evident that we are dealing with points of intersection of C^3 and K^6, not lying at the same time on K^2, so these are the six points (A_{12}, A_{34}), (B_{12}, B_{34}), (C_{12}, C_{34}) formerly found.

12 It is now evident, that the curve K^6 has ten double points; so it is unicursal. Of these points six lie on C^3, the remaining 4 are 1, 2, 3, 4. The six double points representing 12 points of intersection of K^6 with C^3, there are still 6 points; these are evidently the points where C^3 also intersects K^2, so that now all the points of intersection of C^3 and K^6 are found.

Moreover it is clear that the curve K^6 touches the curve K^2 in the six common points, so that it has no more points in common with it.

An additional remark is, that the 10 double points have a particular position in reference to each other. They are situated so, that the points A lie in pairs with the 4 points 1, 2, 3, 4 on a conic. This corresponds with the geometrical truth that the ten double points of a curve of the 6th order cannot have an arbitrary position in reference to each other.

13. The algebraic reckoning comes to a similar result. Let the hyperboloid be:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cdot \frac{z^2}{c^2} = 1\,.$$

This is rectangular, if

$$\frac{1}{a^2} + \frac{1}{c^2} = \frac{1}{b^2} \quad \text{or} \quad \frac{1}{b^2} - \frac{1}{c^2} = \frac{1}{a^2}\,.$$

If we start from a general equation of a quadratic surface

$$a_{11}\,x^2 + a_{22}\,y^2 + a_{33}\,z^2 + \ldots\ldots = 0\,,$$

then as is known $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$ are given as roots of the equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0 ,$$

or

$$\lambda^3 + 3A\lambda^2 + 3B\lambda + C = 0 ,$$

so that the condition, which must be satisfied by the 3 roots λ_1, λ_2, λ_3, is

$$\lambda_1 + \lambda_2 = \lambda_3 .$$

Now we have $\lambda_1 + \lambda_2 + \lambda_3 = -3A$, $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 3B$, $\lambda_1\lambda_2\lambda_3 = -C$, from which results after some deduction as a relation between the coefficients

$$27A^3 - 36AB + 8C = 0 .$$

Expressed in the coefficients of the general equation this becomes

$$27(a_{11}+a_{22}+a_{33})^3 -$$

$$-36(a_{11}+a_{22}+a_{33})\left(a_{11}a_{22}+a_{22}a_{33}+a_{33}a_{11}-a_{12}^2-a_{23}^2-a_{31}^2\right)+$$

$$+8(a_{11}a_{22}a_{33}) = 0 ;$$

so we see, that this is a relation where the coefficients appear in the third order.

If there is a pencil of quadratic surfaces of the second order, we substitute $a_{11} + kb_{11}$ for a_{11}; for k we obtain a cubic equation, which proves that there are three rectangular hyperboloids in the pencil, a result corresponding with that of the geometrical considerations.

Up till now the treatment of the problem has borne a general character. For a complete insight the imaginary circle at infinity must be exchanged for the arbitrary conic K^2; there are moreover many particular cases. This would however lead to too extensive discussions; so this communication must be concluded here.