KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday January 27, 1900.

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The following papers were read:

Astronomy. — "*The determination of the Apex of the Solar motion.*" By Prof. J. C. KAPTEYN.

1. *Fundamental hypothesis.*

It is usually taken for granted that the best known determinations of the direction of the solar motion in space are based on the following hypothesis:

Hypothesis H. The peculiar proper motions of the fixed stars have no preference for any particular direction.

In reality however this is not quite correct; at a closer investigation we find that neither the method of AIRY, nor that of ARGELANDER is entirely based on this hypothesis, and yet these two were almost without exception used in all modern determinations of the position of the Apex.

Finding the distribution of the proper motions in regard to the great circles drawn through the position of the Apex as determined by these methods unsatisfactory, KOBOLD concludes that we must drop the hypothesis H, as not agreeing sufficiently with the facts. Such a conclusion however cannot be accepted until at least one computation has been made, which is completely founded on this hypothesis.

This has given rise to the following investigation, in which I have tried to develop a method satisfying this condition. After having explained the method I shall give a short criticism of those of AIRY, ARGELANDER and KOBOLD, which criticism however lays no claim whatever to being complete.

2. *Meaning of the letters and simple relations.*

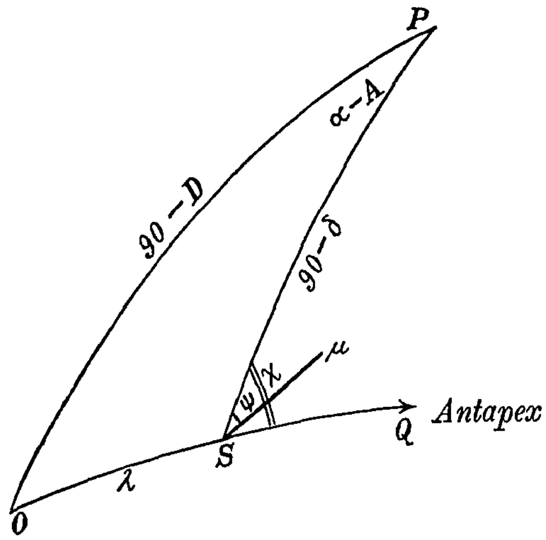

Fig. 1.

O the Apex;
P the Pole of the Equator;
S an arbitrary Star;
A and D the coordinates of the Apex O;
α and δ the coordinates of the Star S;
$\lambda = OS =$ the angular distance of the Star to the Apex;
$\mu = S\mu$ the observed motion of S;
SQ the direction towards the Antapex = direction of the parallactic proper motion;

$\psi = PS\mu$ = angle made by the total proper motion μ with the declination circle;
$\chi = PSQ$ = angle made by parallactic proper motion with th e declination circle;
v = component of total proper motion in direction SQ;
τ = component at right angles to the former (sign same as that of $sin(\chi-\psi)$);
(1) $p = \chi - \psi = \mu SQ$ = angle made by μ with the parallactic proper motion;
h = linear motion of the solar system in space;
ϱ = distance of the star to the solar system.

We have:

(2) $\frac{h}{\varrho} \sin\lambda$ = parallactic proper motion of star S;

Moreover let:

ν (fig. 2) = peculiar proper motion of S;
α' the angle made by this peculiar proper motion with the parallactic.

We then have the following relations:

$$(3) \quad v = \mu \cos(\chi - \psi) = \mu \cos p$$

$$(4) \quad \tau = \mu \sin(\chi - \psi) = \mu \sin p$$

$$(5) \quad \begin{cases} \dfrac{\partial v}{\partial A} = -\tau \dfrac{\partial \chi}{\partial A} \\ \dfrac{\partial v}{\partial D} = -\tau \dfrac{\partial \chi}{\partial D} \end{cases}$$

$$(6) \quad \begin{cases} \dfrac{\partial \tau}{\partial A} = v \dfrac{\partial \chi}{\partial A} \\ \dfrac{\partial \tau}{\partial D} = \nu \dfrac{\partial \chi}{\partial D} \end{cases}$$

3. *Stars in a very limited part of the sky.* First we consider only a group of stars lying so close together that we can practically assume that they are all at the same point of the sky.

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We have to express that these stars satisfy the hypothesis *H*.

The very first condition, the one which I shall here use exclusively, resulting from this hypothesis is this, that the sum of the projections of the peculiar proper motions on any direction is equal to *zero*.

This can also be expressed as follows: the resultant of the peculiar proper motions must be equal to *zero*.

If we project the proper motion ν (fig. 2) on

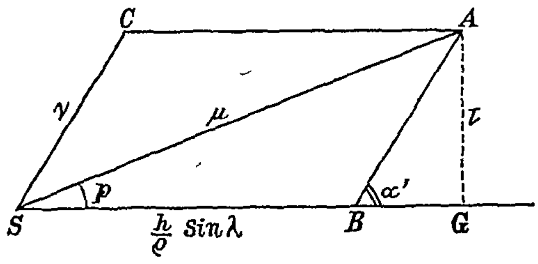

Fig. 2.

the direction SG to the Antapex and at right angles to it, this condition is expressed by the equations:

$$(7) \quad \Sigma\, \nu \cos\alpha' = 0 \qquad\qquad \Sigma\, \nu \sin\alpha' = 0\ .$$

Now along with the peculiar proper motion each star has a parallactic motion $\frac{h}{\varrho}\sin\lambda$, in the direction of SG to the Antapex. Consequently the entire proper motion μ of each star has for components

$$v = \nu\cos\alpha' + \frac{h}{\varrho}\sin\lambda \qquad \tau = \nu\sin\alpha'$$

so that if we take the sum for all the stars of the group we obtain, according to the conditions (7).

$$(8) \quad \Sigma\, v = \sin\lambda\, \Sigma\, \frac{h}{\varrho} \qquad \Sigma\, \tau = 0.$$

The resultant of all the total proper motions μ is directed along SG, from which we conclude at once that the sum of the projections of the motions μ on this direction is greater than on any other. If therefore the direction in which the Antapex is situated as seen from the group of stars under consideration is unknown, it can be derived from the observed proper motions by the fact that for this direction

(9) Σv must be maximum.

It is easy to see that for a group of stars as that here considered, this condition is equivalent to the second condition of (8)

(10) $\Sigma \tau = 0$

This latter however is not so easily extended to all parts of the sky.

4. *Influence of the different distances of the stars.* For a group of stars at one point of the sky it is easy to derive from condition (9) or (10) the direction to the Antapex. We can arrive at a result however, at least theoretically more accurate, by modifying that condition (without throwing up the principle).

It is easy to see at once that if we have stars of greatly differing distances to the sun, the stars whose distance is very large (and whose proper motion is therefore as a rule very small) will influence the result much less than stars at a smaller distance (and with as a rule greater proper motion).

If we start from the principle that one and the same irregularity in the distribution of the linear, peculiar proper motions must have the same effect on the accuracy of the direction to the Antapex for stars at a greater distance and for those at a smaller distance, it is easy to show that not the condition (9) but

(11) $\Sigma \varrho v$ maximum.

must be satisfied.

This one would certainly be preferable to (9), if the distances of the stars were known. This being however the case for so very few objects we are forced to adhere to the theoretically less valuable condition (9).

Fortunately however the objection, arising from the predominating influence of the stars nearest to the sun, may be met to a great extent.

5. *Grouping according to amount of proper motion.*

This may be done by grouping the stars into classes included between pretty narrow, determined limits of the proper motion. The separate results for these groups will finally be combined into a single one, taking due account of their probable errors. We will first have to show however that for such groups of stars the condition (9) is still satisfied, for from considerations as were given in Astr. Nachr. N°. 3487 Page 100 etc. (and which for the sake of brevity we must omit here) follows, that, for groups as are meant here, the distribution of the peculiar proper motions will *certainly no longer satisfy Hypothesis H.*

This is easily done. For it is seen at once that, whereas the former of the conditions (7) ceases in general to be satisfied by such groups, the condition

$$\Sigma \nu \sin \alpha' = 0$$

will still hold. We immediately conclude to this from reasons of symmetry. Now as $\nu \sin \alpha' = \tau$, we shall also still have $\Sigma \tau = 0$ and the condition (9) will be satisfied which is equivalent to it.

Consequently there can be no objection to the grouping in classes of determined proper motions. With this the only advantage which condition (11) might have over (9) disappears in a great measure. In what follows we shall therefore entirely neglect the conditions (11).

6. *Stars scattered about the whole sky or any considerable part of it.*

So every region of the sky gives a condition of the form (9). These might all be combined to *one* condition

$$(12) \quad \Sigma v \text{ maximum}$$

where the sum is to be extended to the stars available in all parts of the sky. In that way however not the most *accurate* determination of the position of the Apex will be obtained.

To arrive at a more advantageous combination the following problem is to be solved:

Given that for the various parts of the sky the accidental deviations from Hypothesis *H* are equal, to combine the conditions (12), which hold for the separate zones of constant λ, in such a way that the effect of those deviations on the coordinates of the Apex which have to be determined, be a minimum.

The solution of this problem which gives rise to no particular difficulties, shows that the Σv of each region must be multiplied by its corresponding value of $\sin \lambda_0$ before they are combined into a single sum.

Consequently for the whole sky we shall *not* have to satisfy the condition (12), but

$$(I) \quad \Sigma v \sin \lambda_0 \text{ maximum.}$$

Second form of the method.

As has already been remarked the objection, that by using (I) the large proper motions exercise a very predominant influence, may be avoided by a grouping into classes of different proper motion. This can be done in still another way.

By substituting the value (3) for v in (I) it becomes

$$\Sigma\, \mu \cos p \sin \lambda_0 \text{ maximum.}$$

As this holds also for stars whose proper motion is included between determined limits, it also holds for stars with absolutely the same proper motion $\mu = \mu_1$. For such a group the condition becomes

$$\text{(II)} \quad \Sigma \cos p \sin \lambda_0 \text{ maximum}$$

and as each value of the proper motion leads to such a condition, it must also be satisfied by all the stars together.

The equations for the coordinates of the Apex, obtained in this way contain only the *directions* and are entirely independent of the *amount* of the proper motion.

It seems to me however that the condition (I), at least if it is applied to stars whose proper motions are included between pretty narrow limits, is preferable to (II) especially for this reason, that the former is a more direct consequence of hypothesis H on which the investigation is based.

8. *Derivation of the Apex from the condition* (I).

To determine the coordinates of the Apex in such a way that condition (I) is satisfied, the differential quotients in regard to A and D of $\Sigma\, v \sin \lambda_0$ must disappear. Consequently we have with the aid of (5)

$$\text{(13)} \quad \Sigma\, \tau \frac{\partial \chi}{\partial A} \sin \lambda_0 = 0 \qquad \Sigma\, \tau \frac{\partial \chi}{\partial D} \sin \lambda_0 = 0$$

which for stars at *one* point of the sky is reduced to the *single* equation $\Sigma\, \tau = 0$, as of course was to be expected.

Let A_0 and D_0 be approximate values of A and D and dA, dD the required corrections of these. All the quantities computed with the aid of these approximate values will be distinguished by means of an appended $_0$.

So v_0 and τ_0 will represent the projections of the proper motion μ on the great circle through the star and the approximate position of the Apex and at right angles to it.

We thus have in the equation (13)

$$\tau = \tau_0 + \left(\frac{\partial \tau}{\partial A}\right)_0 dA + \left(\frac{\partial \tau}{\partial D}\right)_0 dD = \tau_0 + v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

$$\frac{\partial \chi}{\partial A} = \left(\frac{\partial \chi}{\partial A}\right)_0 + \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 dA + \left(\frac{\partial^2 \chi}{\partial A\, \partial D}\right)_0 dD$$

Evidently these equations hold only as long as we do not approach the Apex or the Antapex within distances of the order of dA and dD where the terms of a higher order may not be neglected. It will be best therefore to exclude entirely the stars close to the approximate position of the Apex. This cannot cause any considerable loss of weight. I find e. g. that of the stars of BRADLEY only a fourteenth part have $\sin \lambda < 0.40$ and less than one eighth part have $\sin \lambda < 0.50$.

The first of the equations (13) now becomes

$$dA\, \Sigma \left\{ v_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2 + \tau_0 \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 \right\} \sin \lambda_0 +$$

$$+ dD\, \Sigma \left\{ v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 + \tau_0 \left(\frac{\partial^2 \chi}{\partial A \partial D}\right)_0 \right\} \sin \lambda_0 = - \Sigma\, \tau_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \sin \lambda_0$$

The quantities τ must be in all parts of the sky as often *positive* as *negative*. According to what we have discussed this is an immediate consequence of the hypothesis H (compare form. (10)).

So $\Sigma\, \tau \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0$ will already disappear for limited parts of the sky. The same holds *à fortiori* for the sum extended over the whole sky.

$\Sigma\, \tau_0 \left(\frac{\partial^2 \chi}{d A^2}\right)_0$ differs from the preceding sum only in the quantities τ being computed with an approximate Apex, the coordinates of which need still the corrections dA and dD. This quantity will thus be of the order of dA and dD and may be neglected in the coefficient of dA. The same holds for all quantities containing τ_0 in the coefficients of dA and dD. So the above equation is reduced to the former of the two following ones, in which the sums are indicated with the notation used in the theory of least squares:

$$(14) \begin{cases} \left[v_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2 \right] dA + \left[v_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \right] dD = - \left[\tau_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \right] \\ \left[v_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \right] dA + \left[v_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0^2 \right] dD = - \left[\tau_0 \sin\lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \right] \end{cases}$$

The second of these equations is derived in quite the same way as the first.

9. *Derivation of the Apex from condition* (II).

The maximum conditions are:

$$\Sigma \sin p \sin \lambda_0 \frac{\partial \chi}{\partial A} = 0 \qquad \Sigma \sin p \sin \lambda_0 \frac{\partial \chi}{\partial D} = 0 \ ,$$

in which we put:

$$\sin p = \sin p_0 + \cos p_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + \cos p_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

$$\frac{\partial \chi}{\partial A} = \left(\frac{\partial \chi}{\partial A}\right)_0 + \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 dA + \left(\frac{\partial^2 \chi}{\partial A\, \partial D}\right) dD \ .$$

The first of the equations thus becomes:

$$dA\, \Sigma \left\{ \cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2 + \sin p_0 \sin \lambda_0 \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 \right\} +$$

$$+ dD\, \Sigma \left\{ \cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 + \sin p_0 \sin \lambda_0 \left(\frac{\partial^2 \chi}{\partial A\, \partial D}\right)_0 \right\}$$

$$= - \Sigma \sin p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 .$$

Here again, for quite similar reasons as in the equations of the preceding article, the terms with $\sin p_0$ in the coefficient of dA and dD may be neglected, they being of the order of dA and dD. The equation is thus reduced to the former of the two following ones (the second is found in the same way as the first):

$$(15) \left\{ \begin{aligned} \left[\cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2\right] dA + \left[\cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0\right] dD &= - \left[\sin p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0\right] \\ \left[\cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0\right] dA + \left[\cos p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0^2\right] dD &= - \left[\sin p_0 \sin \lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0\right] \end{aligned} \right.$$

10. AIRY's *method.*

In his derivation of the position of the Apex and the amount of the linear proper motion of the sun, AIRY starts from the idea, that the peculiar proper motions having no preference for any particular directions, may be treated entirely as errors of observation.

Hence each star gives two equations of condition between h, A and D, expressing that the observed proper motions projected on two mutually perpendicular directions are equal to the projections on those same directions of the parallactic proper motions. AIRY chooses for the two directions the parallel and the declination circle.

To get a clear insight into the *character* of AIRY's solution it is preferable however to choose for these directions the direction of the star towards the Antapex and the great circle through the star at right angles to the former.

Doing this his equations of condition get the form

$$(16) \quad \tau = 0$$

and

$$(17) \quad v = \frac{h}{\varrho} \sin \lambda \, .$$

So we can say that by AIRY's method A, D and h are determined in such a way that all the equations (16) and (17) are satisfied in the best way possible. Now as AIRY and every one who has applied his method, have solved these equations by least squares, this determination comes in reality to the choosing of A, D and h in such a way that both

$$(18) \quad \Sigma \tau^2 \text{ minimum}$$

$$(19) \quad \Sigma \left(\frac{h}{\varrho} \sin \lambda - v\right)^2 \text{ minimum.}$$

The former of these does not contain the unknown quantity h and only leads to a determination of A and D. The second gives the three unknown quantities, so that we arrive at two independent determinations of A and D and one of h. I will here discuss the two conditions (18) and (19) separately.

11. *The condition $\Sigma \tau^2$ minimum.*

The minimum conditions are (with the aid of (6)):

$$(20) \quad \Sigma \tau v \frac{\partial \chi}{\partial A} = 0$$

$$(21) \quad \Sigma \tau v \frac{\partial \chi}{\partial D} = 0 \, .$$

For stars all situated at the same point of the sky, they are reduced to this one

$$(22) \quad \Sigma \tau v = 0$$

differing from the condition

$$\Sigma \tau = 0$$

which we have found as a necessary consequence of hypothesis *H*. This proves sufficiently that in reality AIRY's method (at least if his equations of condition are treated with least squares) does not agree with hypothesis *H*.

A few examples will show this still more clearly and will at the same time prove that the application of conditions (9) and (18) may lead to *very considerably* different solutions.

1st *example* (see fig. 3). In each of the two points of the celestial

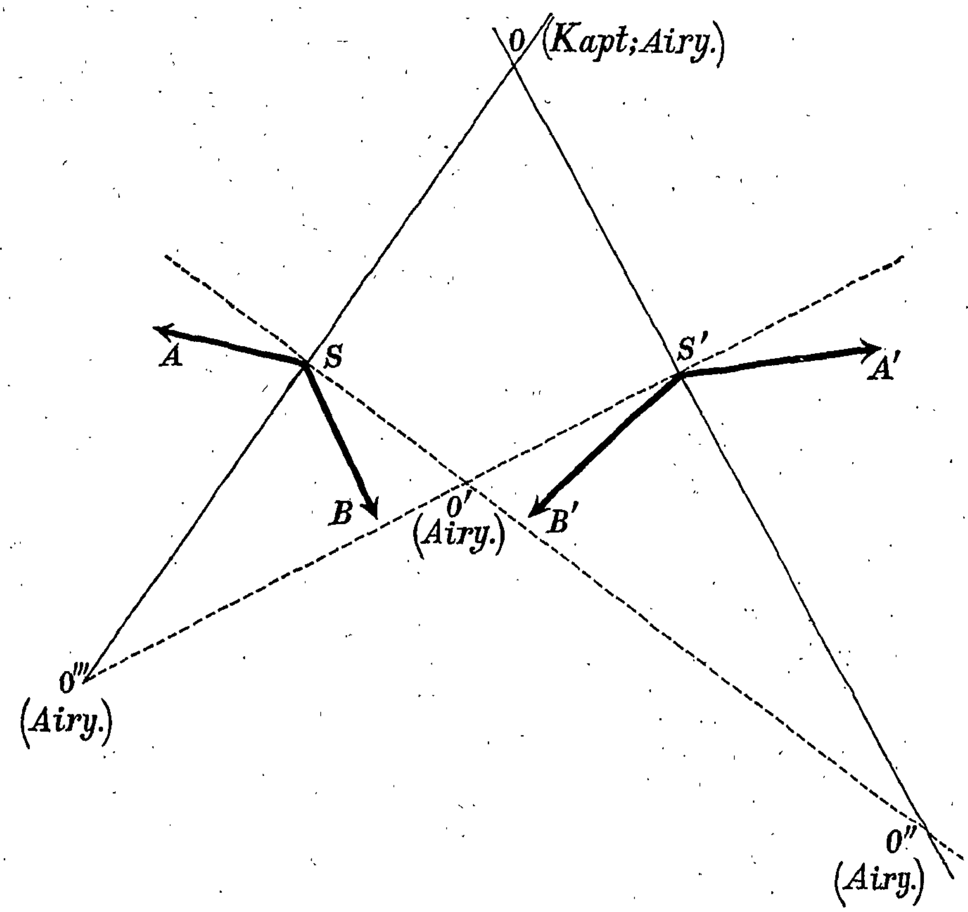

Fig. 3.

sphere *S* and *S'* we find two stars. The proper motions *SA* and *SB* of the two stars in *S* are equal and we will begin by assuming that their directions form an obtuse angle. The same holds for the stars in *S'*.

We see at once that the direction which causes the $\Sigma \tau$ [1]) of the proper motions of the stars in S to disappear is the bisectrix SO of the angle ASB. Likewise the line which in S' makes $\Sigma \tau = 0$ will be the bisectrix $S'O$ of $A'S'B'$.

From the given proper motion we conclude therefore, according to the method proposed by me, to a position O of the Apex.

On the other hand the direction which makes minimum the $\Sigma \tau^2$ of the proper motions of the stars in S is evidently the line SO' at right angles to the bisectrix; likwise in S' the line satisfying this condition is the line $S'O'$ at right angles to $S'O$.

According to the condition $\Sigma \tau^2$ min. of Airy we conclude therefore to a position for the Apex (or Antapex) in O'.

If we leave the angle ASB unchanged, but reduce $B'S'A'$ in such a way, that the bisectrix does not change place, then in the moment that that angle passes through 90° the Apex (Antapex) according to Airy's determination, will suddenly leap over from O' to O'', where it remains when the angle $B'S'A'$ is still more reduced. If the angle $B'S'A'$ had retained its original value ($>$ 90°) and angle ASB had been reduced in the above stated way, the Apex would have leaped from O' to O'''. If then again we had reduced $A'S'B'$, at the moment of this angle passing through 90°, the Apex (Antapex) would have leaped from O''' to O.

In the nature of the problem there seems to be no reason whatever for such leaping [2]) and in our determination the Apex remains where it is notwithstanding the changes introduced.

Moreover it seems very little plausible indeed to assume for the Apex one of the positions O', O'', O'''.

The place determined according to both methods coincides evidently only when both angles are acute.

2nd example (see fig. 4). For stars in the region S let the line

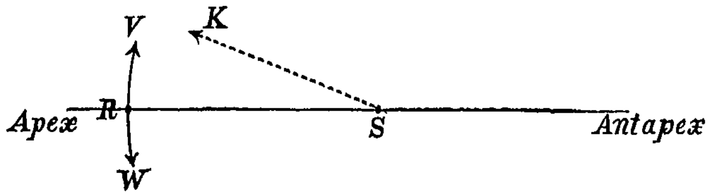

Fig. 4.

[1]) According to the statement in § 3 the condition for stars at one point of the sky is equivalent to $\Sigma \tau = 0$.

[2]) There are still many other cases in which the condition (18) leads to discontinuous changes in the Apex for continuous changes of the data of the problem.

to the Apex be determined by a number of proper motions (not shown in the figure) which, to avoid complication, we shall suppose to be all direct. If now one more star be added, whose proper motion SK makes an acute angle with the line towards the Apex (which therefore is retrograde) we easily see that according to the condition $\Sigma\tau^2$ min. (AIRY) the line SR towards the Apex will have to be turned somewhat more into the direction RV, whilst the condition (10) demands a movement in the direction RW[1]).

12. *The condition* $\Sigma\left(\frac{h}{\varrho}\sin\lambda - v\right)^2$ *minimum.*

The equations of condition are of the form

$$(23)\quad \frac{h}{\varrho}\sin\lambda = v$$

They contain the distances which are as a rule unknown. This is certainly the chief objection to the use of these equations. They seem therefore much more suitable to give information about mean parallaxes of definite groups of stars when once the Apex is known, than to assist in determining the position of the Apex itself.

For the calculations according to AIRY's method different ways have been followed to escape the difficulty arising from the unknown distances. One of the commonest practices (STUMPE, PORTER, etc.) is to divide the stars into groups included between narrower or wider limits of proper motion and then to assume the distance of the stars of each group to the sun to be the same.

If this be true in the mean of great numbers of stars for different parts of the sky, it might seem for a moment that we might really derive trustworty values of dA, dD and the mean value of $\frac{h}{\varrho}$ from a treatment of the equations (23). Meanwhile we must bear in mind that at all events a new hypothesis has been introduced, viz., that the mean parallax of stars with equal proper motion in different parts of the sky, is the same. If this is not the case the position obtained for the Apex too will be in general erroneous.

However there is another and decisive objection to the use of equations (23) if we have grouped the stars according to their proper

[1]) A practical advantage of our method over AIRY's may still be mentioned here: In AIRY's method the large proper motions have a much more predominant influence on the results even than in ours. This is easy to see from the normal equations to be given in art. 15.

motions, viz: that these equations for groups of stars included between determined, arbitrary limits of the proper motion, howsoever numerous the stars may be, *are certainly in general not true*[1]).

This is evident from the argument given in Astron. Nachr. N°. 3487, pages 100—102, to which we must refer here. The error committed will certainly be *different* in general for regions with different λ even in the case that the proper motions are equal.

So not only do derivations such as those of STUMPE (Astron. Nachr. N°. 3000) and many others, give entirely illusory determinations for the secular parallax of the stars (as I already tried to show in Astr. Nachr. N°. 3487) but neither can the determination of the position of the Apex be defended. It may even be anticipated with great probability that the error must change systematically with the amount of the proper motion, so that the regular change found by STUMPE in the declination of the Apex for his various groups has nothing particularly surprising.

Other writers as i. a. L. STRUVE attribute determined parallaxes to stars of determined magnitude. The last decisive objection disappears here, but not the first. It runs as follows: we assume that at least the mean parallax of the stars of determined magnitude is everywhere in the sky the same. For the galaxy and outside it I have already tried to show, some years ago, (Verslag Kon. Akad. Jan. 1893) that this is probably *not* the case

To sum up, according to the preceding, AIRY's method comes to the determination of the coordinates of the Apex and the linear motion of the sun in such a way that the conditions (18) and (19) are satisfied.

The first condition does not contain the distances but does not in general satisfy the conditions $\Sigma\,\tau = 0$ for stars in one and the same part of the sky, which must be considered as the principal condition derivable from the hypothesis H. The second condition contains the distances which are in general unknown. This causes the introduction of hypotheses which are more or less probable, and which may easily exercise an injurious influence on the determination of A and D. Particularly the grouping according to proper motion must be absolutely objected to in the application of AIRY's method, because implicit suppositions are introduced which are *certainly* not realized.

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[1]) It is even not permissable to exclude stars with very small proper motion.

13. *Method of* Argelander.

In this method each star gives an equation of condition of the form

$$(24) \quad p = 0 \qquad (\text{weight } sin^2 \lambda_0)$$

They are treated with least squares. So in reality A and D are determined by the condition

$$(25) \quad \Sigma p^2 sin^2 \lambda_0 \text{ minimum,}$$

giving the minimum conditions

$$(26) \quad \Sigma p \, sin^2 \lambda_0 \frac{\partial \chi}{\partial A} = 0 \qquad \Sigma p \, sin^2 \lambda_0 \frac{\partial \chi}{\partial D} = 0.$$

For a single region of the sky the two are reduced to this *one*

$$(27) \quad \Sigma p = 0,$$

so that here neither the condition furnished by hypothesis H is satisfied.

The objection to the method of Argelander consists chiefly in this that the retrograde proper motions have too great an influence.

Let for instance the proper motions μ_1 μ_2 μ_3 μ_4 (belonging to stars in the same region of the sky) make with an assumed direction towards the Antapex angles of +20°, +10°, —10°, —20°. As long as we know only these proper motions the assumed direction towards the Antapex, both according to my method and to that of Argelander, will be the most probable. If however a proper motion μ_5 is added, making with the assumed direction towards the Antapex an angle of 170°, this direction, according to Argelander's method will have to be corrected by 34°, whereas according to our method that correction will be only 2°.1 in the same direction. Since long it has been remarked moreover that in Argelander's method too, discontinuous changes in the place of the Apex may be caused by continuous changes of the proper motions.

The following example will prove this clearly.

In a definite region of the sky there are n stars whose proper motion is in perfectly the same direction. This common direction is assumed as the approximate direction towards the Antapex. We now add *one* star, making with that direction the angle

$$p_0 = 180 - \omega$$

where ω is a very small quantity. If this is neglected, it follows from (27) that the direction towards the Antapex has to be corrected by

$$-\frac{180^\circ}{n+1}$$

If however for the added star we had

$$p_0 = 180 + \omega = -(180 - \omega)$$

then we should have found for that correction

$$+\frac{180}{n+1}.$$

So there is a leap of $\frac{360^\circ}{n+1}$.

There is again no foundation for such a leap in the nature of the problem, and it does not appear in our solution.

14. *Method of* KOBOLD *(Bessel).*

I need but say a few words of this method, as KOBOLD himself clearly states that his method is not based on hypothesis *H.*

He determines the Apex of the motion of the sun in such a way that the great circle of which the Apex is the pole, approaches as closely as possible to the pole of the proper motions of all the stars.

To satisfy this condition he makes

$$\Sigma \cos^2 Q \text{ minimum},$$

where Q represents the distance from the Apex of the pole of a proper motion. Expressed in the quantities used by us, the condition is

$$(28) \quad \Sigma \sin^2 \lambda \sin^2 p \text{ minimum}.$$

This is satisfied if we write down for each star an equation of condition

$$(29) \quad \sin \lambda \sin p = 0$$

and then solve the whole of these equations with least squares.

This method cannot be tested by the condition (10). It is namely a peculiarity of this method, that whereas, according to the other methods, from stars of one part of the sky, only a *direction* can be derived, in which the Apex must be situated, we find by KOBOLD's method a *complete* determination of the position of that point.

Its position is no other than that of the group of stars itself. In the choice of the position of the Apex each region votes as it were for itself. Every line passing through this region thus passes through the Apex too, so that at the same time the condition (10) is satisfied and it is not.

This peculiarity of the method together with this second (which exists for stars of *one* part as well as for stars of *all* parts of the sky) that for the direction of the motion of an arbitrary number of stars we may substitute a diametrically opposite motion without the slightest effect on the coordinates of the Apex, appears to me sufficient to declare the method unsuitable *for the determination of the direction of the motion of the sun.*

15. *Abridged calculation.*

It is a very common practice in the derivation of the coordinates of the Apex, to abridge the work of computation by taking the mean of the proper motions of a greater or smaller number of stars situated close together. I wish to point out that in this way the result, derived by means of the various methods, will approach in general to those which will be found by the method proposed here. So, far from having been more or less impaired by this abridged calculation, the results must have gained considerably in accuracy.

It must be borne in mind however that in this way, in all methods except in that proposed by me, the principle is sacrificed, at least in part.

The proof of what has been advanced here will be best given by writing out in full and in a similar form for the various methods, the equations of condition and the normal equations ensuing from these. I begin by giving them.

a. Method of AIRY (as modified).

I leave out of consideration condition (19), this being the only one dependent on the distances.

As

$$\tau = \tau_0 + \left(\frac{\partial \tau}{\partial A}\right)_0 dA + \left(\frac{\partial \tau}{\partial D}\right)_0 dD = \tau_0 + v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

the equations (16) become

$$(30) \quad v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD = -\tau_0$$

which, treated with least squares, give the normal equations:

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$$(31)\quad \begin{cases} \left[v_0^2\left(\frac{\partial \chi}{\partial A}\right)_0^2\right] dA + \left[v_0^2\left(\frac{\partial \chi}{\partial A}\right)_0\left(\frac{\partial \chi}{\partial D}\right)_0\right] dD = -\left[\tau_0\, v_0\left(\frac{\partial \chi}{\partial A}\right)_0\right] \\ \left[v_0^2\left(\frac{\partial \chi}{\partial A}\right)_0\left(\frac{\partial \chi}{\partial D}\right)_0\right] dA + \left[v_0^2\left(\frac{\partial \chi}{\partial D}\right)_0^2\right] dD = -\left[\tau_0\, v_0\left(\frac{\partial \chi}{\partial D}\right)_0\right] \end{cases}$$

They are of course identical with the equations (20) and (21) if in the reduction of these we treat the quantities $\frac{\tau_0}{v_0}$ as quantities of the order of dA and dD.

b. Method of ARGELANDER.

If we reduce to unity of weight, the equations of condition (24) may be writen

$$(32)\quad p \sin \lambda_0 = 0$$

or by writing

$$p = p_0 + \left(\frac{\partial p}{\partial A}\right)_0 dA + \left(\frac{\partial p}{\partial D}\right)_0 dD = p_0 + \left(\frac{\partial \chi}{\partial A}\right)_0 dA + \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

$$(33)\quad \sin \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + \sin \lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD = -p_0 \sin \lambda_0\,,$$

which lead to the normal equations:

$$(34)\quad \begin{cases} \left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial A}\right)_0^2\right] dA + \left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial A}\right)_0\left(\frac{\partial \chi}{\partial D}\right)_0\right] dD = -\left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial A}\right)_0 p_0\right] \\ \left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial A}\right)_0\left(\frac{\partial \chi}{\partial D}\right)_0\right] dD + \left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial D}\right)_0^2\right] dD = -\left[\sin^2\lambda_0\left(\frac{\partial \chi}{\partial D}\right)_0 p_0\right] \end{cases}$$

which again will be identical with (26), if we treat the quantities p_0 as of the order of dA and dD.

c. Method of KOBOLD.

By introducing

$$\sin \lambda = \sin \lambda_0 + \cos \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \cos \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0 dD$$

$$\sin p = \sin p_0 + \cos p_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + \cos p_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

the equations of condition (29) become

(35) $\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial A}\right)_0 \right\} dA + \left\{ cos\lambda_0 \, sinp_0 \left(\frac{\partial\lambda}{\partial D}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial D}\right)_0 \right\} dD = - sin\lambda_0 \, sinp_0$

which give the normal equations:

(36)
$$\left\{\begin{aligned}
&\left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial A}\right)_0 \right\}^2 \right] dA + \\
&+ \left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial A}\right)_0 \right\} \left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial D}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial D}\right)_0 \right\} \right] dD \\
&\qquad = - \left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial A}\right)_0 \right\} sin\lambda_0 \, sinp_0 \right] \\
&\left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial A}\right)_0 \right\} \left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial D}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial D}\right)_0 \right\} \right] dA + \\
&\qquad + \left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial D}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial D}\right)_0 \right\}^2 \right] dD \\
&\qquad = - \left[\left\{ cos\lambda_0 sinp_0 \left(\frac{\partial\lambda}{\partial D}\right)_0 + sin\lambda_0 cosp_0 \left(\frac{\partial\chi}{\partial D}\right)_0 \right\} sin\lambda_0 sinp_0 \right]
\end{aligned}\right.$$

Let us now assume, as was supposed above, that we take the mean of the proper motions of the stars situated closely together and continue working with these as if they were real proper motions. The effect of thus taking means of a considerable number of motions will of course be, that the peculiar motion, which takes place in various directions, is eliminated for the greater part, so that the mean proper motion found will, with some approximation, represent the mean parallactic motion for the region under consideration.

If we distinguish the values obtained by taking means by dashes over the letters, then we will evidently have for the various regions, with greater or smaller approximation (cf. (8)):

$$\bar{v} = \overline{\frac{h}{\varrho} sin\,\lambda} \qquad \bar{\tau} = 0$$

and consequently

$$tg\, p = \frac{\bar{\tau}}{\bar{v}} = 0$$

If first we take only zones of constant λ_0 and if farthermore we assume that the mean secular parallax $\frac{h}{\varrho}$ is, with some approxi-

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mation, the same for stars in various parts of the sky, we shall see at once that the effect of taking the mean for various parts of the sky in such a zone of constant λ_0, is this, that the different $\bar{v}$'s will become equal with some approximation, whilst moreover for such a zone, as indeed for the whole sky, $\bar{\tau}$ and $\bar{p}$ become *small* quantities.

If therefore we introduce into the equations (31) of AIRY:

$$(37) \quad \begin{cases} \lambda = \text{constant} \\ v_0 = \bar{v} = \text{constant} \\ \tau_0 = \bar{\tau} \end{cases}$$

they will become

$$\bar{v}^2 \left[\overline{\left(\frac{\partial \chi}{\partial A}\right)^2}\right] dA + \bar{v}^2 \left[\overline{\left(\frac{\partial \chi}{\partial A}\right)\left(\frac{\partial \chi}{\partial D}\right)}\right] dD = -\bar{v}\left[\overline{\tau \frac{\partial \chi}{\partial A}}\right]$$

$$\bar{v}^2 \left[\overline{\frac{\partial \chi}{\partial A}\frac{\partial \chi}{\partial D}}\right] dA + \bar{v}^2 \left[\quad \overline{\left(\frac{\partial \chi}{\partial D}\right)} \quad\right] dD = -\bar{v}\left[\overline{\tau \frac{\partial \chi}{\partial D}}\right]$$

These equations are identically the same as those into which our equations (14) are transformed, if in these too we introduce the values (37).

So zones of the same λ will furnish approximately the same results if treated according to both methods. Hence the combination of all these partial solutions will certainly not lead to strongly deviating results.

A still closer correspondence may be expected between the results of ARGELANDER's method and the results of the second form of that proposed by us, when by taking means, all the angles are first made small.

For if we neglect quantities of the order.

$$p^2\, dA \ , \quad p^2\, dD \ , \quad p^3$$

we may write in the second member of the equations (34) $p_0 = \sin p_0$, so that those equations become, if here again we take a zone of constant λ_0,

$$\sin^2\lambda_0 \left[\overline{\left(\frac{\partial \chi}{\partial A}\right)^2}\right] dA + \sin^2\lambda_0 \left[\overline{\frac{\partial \chi}{\partial A}\frac{\partial \chi}{\partial D}}\right] dD = -\sin^2\lambda_0 \left[\overline{\sin p \frac{\partial \chi}{\partial A}}\right]$$

$$\sin^2\lambda_0 \left[\overline{\frac{\partial \chi}{\partial A}\frac{\partial \chi}{\partial D}}\right] dA + \sin^2\lambda_0 \left[\overline{\left(\frac{\partial \chi}{\partial D}\right)^2}\right] dD = -\sin^2\lambda_0 \left[\overline{\sin p \frac{\partial \chi}{\partial D}}\right]$$

which equations are identical with (15) if we introduce in them the same suppositions.

So, here again we find, that zones of identical λ_0 will lead to approximately the same results in the two methods. What holds for each of the zones separately, must also hold with some approximation for the final results.

For KOBOLD's method the approximation will be somewhat less satisfactory. For here we must neglect terms of the order

$$p\, dA, \quad p\, dD, \quad p^2$$

to gain our end.

If we do this, the equations (36) will become

$$2 \sin^2 \lambda_0 \left[\overline{\left(\frac{\partial \chi}{\partial A}\right)^2}\right] dA + 2 \sin^2 \lambda_0 \left[\overline{\frac{\partial \chi}{\partial A}\,\frac{\partial \chi}{\partial D}}\right] dD = -\, 2 \sin^2 \lambda_0 \left[\overline{\sin p \frac{\partial \chi}{\partial A}}\right]$$

$$2 \sin^2 \lambda_0 \left[\overline{\frac{\partial \chi}{\partial A}\,\frac{\partial \chi}{\partial D}}\right] dA + 2 \sin^2 \lambda_0 \left[\overline{\left(\frac{\partial \chi}{\partial D}\right)^2}\right] dD = -\, 2 \sin^2 \lambda_0 \left[\overline{\sin p \frac{\partial \chi}{\partial D}}\right]$$

which are again identical with our equations (15) if we introduce in them the same suppositions. Zones of identical λ treated according to both methods giving approximately identical results, this must lead here also to pretty nearly the same final results.

The calculations of KOBOLD (Astr. Nachr. N°. 3592) confirm this conclusion. The solution which he makes with mean proper motions is the only one which is in a somewhat tolerable agreement with what others have found, calculating with other methods but also with mean values of the proper motions.

KOBOLD finds	$A = 262°.8$	$D = +16°.5$
L. STRUVE finds	$A = 273°.3$	$D = +27°.3$

After all that has been said the conclusion is pretty obvious that what, perhaps more than anything else, must hinder us in accepting the methods used until now for the derivation of the direction of the solar motion is this: that quantities are treated as small ones, which in reality are *not* small[1]).

[1]) From an utterance of Prof. NEWCOMB I conclude that he too ascribes the deviating result of KOBOLD to the reason here stated.

16. *Values of the differential quotients used in the preceding articles.*

The following formulae may serve for the various differential quotients used in the preceding equations.

(For the meaning of the letters see fig. 1).

$$\frac{\partial \chi}{\partial A} = - \frac{\cos D \cos O}{\sin \lambda}$$

$$\frac{\partial \chi}{\partial D} = - \frac{\cos \delta \sin \chi}{\cos D \sin \lambda}$$

$$\frac{\partial \lambda}{\partial A} = - \cos \delta \sin \chi$$

$$\frac{\partial \lambda}{\partial D} = - \cos O$$

where χ, λ and O are to be computed by

$$\sin \lambda \sin \chi = \sin (\alpha - A) \cos D$$

$$\sin \lambda \cos \chi = \cos (\alpha - A) \cos D \sin \delta - \sin D \cos \delta$$

$$\sin \lambda \sin O = \sin (\alpha - A) \cos \delta$$

$$\sin \lambda \cos O = - \cos (\alpha - A) \cos \delta \sin D + \sin \delta \cos D.$$

A few observations of Prof. JAN DE VRIES and Prof. J. A. C. OUDEMANS were answered by the lecturer.

Mathematics. — "*On twisted quintics of genus unity.*" By Prof. JAN DE VRIES.

1. By central projection a twisted curve of order five and genus unity can be transformed into a plane curve of order five with five nodes. Consequently in each point of space meet *five* chords or bisecants of the twisted curve R_5.

If the centre of projection is taken on R_5 a curve of order four with two nodes is obtained. From this ensues that through each point of R_5 *two* trisecants may pass.

2. The bisecants that meet a given right line l form a surface