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16. *Values of the differential quotients used in the preceding articles.*

The following formulae may serve for the various differential quotients used in the preceding equations.

(For the meaning of the letters see fig. 1).

$$\frac{\partial \chi}{\partial A} = -\frac{\cos D \cos O}{\sin \lambda}$$

$$\frac{\partial \chi}{\partial D} = \frac{\cos \delta \sin \chi}{\cos D \sin \lambda}$$

$$\frac{\partial \lambda}{\partial A} = -\cos \delta \sin \chi$$

$$\frac{\partial \lambda}{\partial D} = -\cos O$$

where χ, λ and O are to be computed by

$$\sin \lambda \sin \chi = \sin (\alpha - A) \cos D$$

$$\sin \lambda \cos \chi = \cos (\alpha - A) \cos D \sin \delta - \sin D \cos \delta$$

$$\sin \lambda \sin O = \sin (\alpha - A) \cos \delta$$

$$\sin \lambda \cos O = - \cos (\alpha - A) \cos \delta \sin D + \sin \delta \cos D.$$

A few observations of Prof. JAN DE VRIES and Prof. J. A. C. OUDEMANS were answered by the lecturer.

Mathematics. — "*On twisted quintics of genus unity.*" By Prof. JAN DE VRIES.

1. By central projection a twisted curve of order five and genus unity can be transformed into a plane curve of order five with five nodes. Consequently in each point of space meet *five* chords or bisecants of the twisted curve R_5.

If the centre of projection is taken on R_5 a curve of order four with two nodes is obtained. From this ensues that through each point of R_5 *two* trisecants may pass.

2. The bisecants that meet a given right line l form a surface

A, on which l is a fivefold line. Ten chords lying in every plane through l the scroll A is of order *fifteen.*

Besides the fourfold curve R_5 the scroll A contains a double curve of which we shall determine the order.

If the points A_i ($i = 1, 2, 3, 4, 5$) lie in a plane with l then the fifteen points $B \equiv (A_i A_k\,,\ A_l A_m)$ belong to the above mentioned curve.

In order to find how many points B are lying on l we assign the point common to l and $A_i A_k$ to the points common to l and the right lines $A_l A_m$, $A_m A_n$ and $A_n A_l$; hereby we create a correspondence (15,15) between the points of l. Two corresponding points only then coincide when a point B lies on l. In the correspondence there are still thirteen other points which differ from B agreeing with such a point; so B represents *two* coincidences. Hence l contains fifteen points B and the above mentioned double curve is of order *thirty.*

3. If l has a point S in common with R_5 then A_{15} breaks up into the quartic cone, with centre S, standing on R_5 and into a surface A_{11}, on which R_5 is a threefold curve, l remaining a fivefold line. Moreover by a very simple deduction it is shown that now the double curve is of order *eight.*

4. If l becomes a bisecant b the surface A_{15} contains two quartic cones. The remaining scroll A_7 has the fourfold line b and the double curve R_5. The double curve (B) disappears here.

By assigning each of the three points of R_5 lying with b in the same plane to the chord connecting the other two, the chords of the scroll A_7 are brought into projective relation with the points of R_5.

So any plane section of A_7 is, just as R_5, of *genus unity* and must have fourteen nodes or an equivalent set of singularities. This curve has five double points on R_5 and a fourfold point on b. Evidently the missing three double points can only be represented by a threefold point derived from a threefold generator of A_7, i.e. from the trisecant of the twisted curve.

So a bisecant will be cut only by one *trisecant.*

5. As b meets in each of its points of intersection with the curve two trisecants, the *trisecants* of R_5 form a *scroll* T_5 of order *five* of which R_5 is a double curve. Evidently T_5 can have no other double curve, so this surface is also of *genus unity.*

Two bisecants meet a trisecant t in each of its points whilst each plane through t contains a chord. All these bisecants form a cubic scroll Δ_3 with double director t. The single director u is evidently a unisecant of R_5. On the scroll Δ_{11} determined by u of course t is a part of the above mentioned double curve.

Each of the double points of the involution determined on u, by the generators of Δ_3 procures coinciding chords; consequently u is the section of two double tangent planes.

6. A conic Q_2 having five points in common with R_5 is not intersected by a trisecant in a point not lying on R_5, for in its points of intersection with R_5 it has ten points in common with T_5. The surface Γ formed by the conics Q_2, the planes of which pass through the line c, is intersected by each trisecant in three points; so Γ is a cubic surface.

The right line c meets five trisecants lying on Γ_3, hence also five bisecants belonging to this surface. As c is intersected by the conic Q_2 of Γ_3 in an involution, there are two conics Q_2 touching it. When c becomes a unisecant then its point S on R_5 is a double point of Γ_3. Besides c still five right lines of Γ_3 pass through S, two of which are trisecants; the remaining three must be bisecants completed to degenerated conics Q_2 by the other trisecants resting on c.

If c becomes a chord, Γ_3 has two double points, each of which supports two bisecants belonging to Γ_3 and two trisecants also lying on the surface. If finally c is a trisecant, Γ_3 becomes the above mentioned surface Δ_3.

So: *All conics Q_2 intersecting two times a given right line form a cubic surface.*

7. *The conics Q_2 passing through any given point P form a cubic surface Π_3 with double point P.*

For only one conic Q_2 passes through P and the point S on R_5, as PS is a single line on the cubic surface Γ_3 determined by PS. From this ensues that R_5 is a single curve of the surface Π_3, so that this is intersected by a trisecant in three points. And as a right line through P has in general with only one conic Q_2 two points in common, one of which is lying in P, P is a double point of Π_3.

On this surface lie the five bisecants meeting in P, moreover the five trisecants by which they are completed to conics. The quadratic cone determined by these five chords intersects Π_3 in a right line p, on which the mentioned trisecants rest; so p has no point in common with R_5. Moreover any given right line through P determining only one conic Q_2 of Π_3, the planes of the conics Q_2 on Π_3 must form a pencil; the planes of the above mentioned degenerated conic Q_2 pass through p, so p is the axis of the pencil. The remaining ten right lines of Π_3 are evidently unisecants of R_5.

8. The axis p determined by P cannot belong to a second surface Π_3, for the five trisecants resting on p determine together with p the bisecants intersecting each other in P.

If P lies on R_5, p is quite undeterminate.

The point P being taken on a trisecant t, through that point two bisecants pass forming with t conics Q_2; the axis p coincides with t, which follows as a matter of course from this, that Π_3 becomes the surface $\varDelta_3$ belonging to t.

9. If P describes the right line a_1, the locus of the axis p is a *cubic scroll* Δ_3, of which a_1 is the linear director. For if P' and P'' are the points common to a_1 and Q_2, then this conic lies on the surface Π_3' and Π_3'' belonging to P' and P''; so its plane contains the corresponding axes p' and p''.

To Δ_3 evidently belong the five trisecants resting on a_1; in the points common to R_5 and these trisecants R_5 is cut by Δ_3. They moreover meet the *double director* a_2 of Δ_3.

These trisecants lie at the same time on the scroll Δ_3' having a_2 as linear director; on this surface a_1 is the double director.

The right lines a_1 and a_2 correspond mutually to one another. If a_1 is itself an axis, each plane through this right line contains only *one* axis p differing from a_1. In that case the surface Δ_3 becomes a scroll of CAYLEY and a_2 coincides with a_1.

In the correspondence (a_1, a_2) each axis is consequently assigned to itself. This also relates to all trisecants, as each of these must be regarded as an axis of each of its points.

10. The five trisecants cut by a_1 and by a_2 also lie on the surface Γ_3 determined by a_1; so this contains the right line a_2 as well.

Therefore both axes p' and p'' lying with a_1 in a plane ω cut each other in the point O common to a_2 and the conic Q_2 determined by ω.

From the mutual correspondence between a_1 and a_2 we conclude that Γ_3 also contains all the conics Q_2, the planes of which pass through a_2. Five bisecants belonging to Γ_3 rest on a_2.

If according to a well known annotation we call the five trisecants consecutively b_3, b_4, b_5, b_6 and c_{12}, then the five bisecants resting on a_1 are indicated by $c_{13}, c_{14}, c_{15}, c_{16}$ and b_{12}, and a_2 meets the bisecants $c_{23}, c_{24}, c_{25}, c_{26}$ and b_1.

It is easy to see that the remaining ten right lines of Γ_3 viz. $a_3, a_4, a_5, a_6, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}$ have each *one* point in common with R_5.

11. Let P be any point of the conic Q_2 meeting a_1 in P' and P''. Now the axes p and p' must intersect each other on Q_2; so p will pass through the point O common to p' and p''.

Consequently the axes p lying in a plane ω pass through a point O of conic Q_2 determined by ω.

As O has been found to describe the line a_2 if ω revolves about a_1, O and ω are focus and focal plane in relation to a linear complex of rays of which a_1 and a_2 are conjugate lines, the axes p and the trisecants t being rays.

12. The conics Q_2 which cut R_5 in P and P' forming a cubic surface, a right line l having α points in common with R_5 meets the $(3-\alpha)$ conics Q_2 through P and P'.

So R_5 is a $(3-\alpha)$-fold curve of the surface Φ, containing the conics Q_2 which pass through P and rest on l. As a trisecant can meet none of those conics in a point not on R_5, Φ is a surface of order $3\,(3-\alpha)$.

Of the $3\,(3-\alpha)$ points common to Φ and the β-secant m $\beta\,(3-\alpha)$ lie on R_5. The remaining $(3-\alpha)\,(3-\beta)$ points of intersection determine as many conics Q_2 resting on l and on m and passing through P as well.

From this we conclude again that all the conics Q_2 cut by l and m will form a surface Ψ, on which R_5 is a $(3-\alpha)\,(3-\beta)$-fold curve. Then however Ψ must be a surface of order $3\,(3-\alpha)\,(3-\beta)$.

If we now notice that a γ-secant n is cut by Ψ in $(3-\alpha)\,(3-\beta)\gamma$ points lying on R_5, thus in $(3-\alpha)\,(3-\beta)\,(3-\gamma)$-points not lying on this curve, it is evident that *three right lines having respectively α, β and γ points common with R_5 determine* $(3-\alpha)\,(3-\beta)\,(3-\gamma)$ *conics Q_2 resting on these lines.*

So any three bisecants meet one conic Q_2 only.

13. Let C_2 be a conic having *no* point in common with R_5.

The surface Π_3, with its double point P on C_2, cuts this curve still in four points P'; consequently C_2 is a fourfold curve of the locus Σ of the conics Q_2, each having two points in common with C_2.

The conic Q_2 lying in the plane Φ of C_2 belongs six times to the section of Σ and Φ.

Moreover as each bisecant of R_5 lying in Φ determines a conic Q_2 of Σ, this surface is of order $4 \times 2 + 6 \times 2 + 10 = 30$.

Through the point S_k of R_5 lying in Φ ten conics Q_2 of Σ_{30}, pass, viz. the four conics determined by the chords $S_k S_l$ and the conic Q_2 to be counted six times containing all the points S_k. So R_5 is a tenfold curve.

If C_2 breaks up into two right lines l and m intersecting each other in P the locus consists of the cubic surface Π_3 belonging to P and the surface Ψ_{27} formed by the conics Q_2 resting on l and m. And now according to 12. the curve R_5 is a ninefold curve of Ψ_{27} and according to 7. a single curve on Π_3; so in accordance with what was mentioned above it is a tenfold curve of $\Sigma_{30} \equiv \Psi_{27} + \Pi_3$.

As C_2 and R_5 have α points in common, we find in a similar way that the conics Q_2 which meet C_2 in two points not situated on R_5 form a surface of order $^3/_2 (4-\alpha)(5-\alpha)$, where R_5 is a curve of multiplicity $^1/_2 (4-\alpha)(5-\alpha)$, C_2 being a $(4-\alpha)$-fold line.

14. We shall still determine the number of conics Q_2 resting on the α-conic C_2, the β-conic D_2 and the γ-conic E_2.

The surface Γ_3 of the conics Q_2, cutting R_5 in P and P', and C_2 have $(6-\alpha)$ points in common. So R_5 is a $(6-\alpha)$-fold curve of the locus of the conic Q_2, passing through P and meeting C_2; so this surface is of order $3(6-\alpha)$.

Of its sections with D_2 a number of $(6-\alpha)(6-\beta)$ are not situated on R_5, which proves that R_5 is a $(6-\alpha)(6-\beta)$-fold curve of the surface of the conics Q_2 resting on C_2 and D_2; so this latter surface is of order $3(6-\alpha)(6-\beta)$.

Consequently there are $(6-\alpha)(6-\beta)(6-\gamma)$ conics Q_2, having a point in common with each of the conics C_2, D_2, E_2.

In particular any three conics Q_2 are cut by one conic Q_2 only.

Physics. — "*The cooling of a current of gas by sudden change of pressure.*" By Prof. J. D. VAN DER WAALS.

If a gas stream under a constant high pressure is conducted through a tube, so wide that we may neglect the internal friction, and this stream is suddenly brought under a smaller pressure, either by means of a tap with a fine aperture, or, as in the experiments of Lord KELVIN and JOULE by means of a porous plug, the