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the corresponding quantities δ and ε [1]) by δ_1 and δ_2 and by ε_1 ε_2, then we have, neglecting the molecules which surround a point immediately:

$$\frac{T_1^4}{T_2^4} = \frac{I_1}{I_2} = \frac{\varepsilon_1^2}{\varepsilon_2^2} = \frac{\delta_1^2 \int \frac{e^{-2\mu_1 r}}{r^2} \frac{y^2+z^2}{r^2} d\tau}{\delta_2^2 \int \frac{e^{-2\mu_2 r}}{r^2} \frac{y^2+z^2}{r^2} d\tau}$$

or

$$\frac{T_1^3}{T_2^3} = \frac{\int \frac{e^{-2\mu_1 r}}{r^2} \frac{y^2+z^2}{r^2} r^2\, dr \sin\varphi\, d\varphi\, d\theta}{\int \frac{e^{-2\mu_2 r}}{r^2} \frac{y^2+z^2}{r^2} r^2\, dr \sin\varphi\, d\varphi\, d\theta} = \frac{\int_0^\infty e^{-2\mu_1 r}\, dr}{\int_0^\infty e^{-2\mu_2 r}\, dr} = \frac{1/\mu}{1/\mu}.$$

Prof. LORENTZ [2]) has deduced, that μ (his quantity α) is inversively proportional to the root of the temperature. And though both the way in which I have arrived at the conclusion that the absorption is inversely proportional to the third power of the temperature, and that in which Prof. LORENTZ found that it is inversely proportional to the root of the temperature, are but rough approximations yet these results differ too much, to attribute this only to the neglections.

Therefore an incorrect assumption must have been made somewhere. And if so I should doubt in the first place the correctness of the assumption, that for all internal motions the increase of the energy must be proportional to the energy of the progressive motion. I should therefore suppose that in collisions there are influences felt which cause the energy of the internal motions, which bring about radiation, to increase more at a rise of the temperature than the energy of the progressive motion of the molecules.

Mathematics. — "*On rational twisted curves*". By Prof. P. H. SCHOUTE.

1. Let P_1, P_2, P_3, P_4, . . . be successive points of a given twisted curve R; then we may consider the centre of circle $P_1 P_2 P_3$ lying in plane $P_1 P_2 P_3$ as well as that of sphere $P_1 P_2 P_3 P_4$. When the

[1]) Proc. Royal Acad. of Sciences, Dec. 1899. Pag. 322.
[2]) Versl. Kon. Akad. v. Wet. April 1898, Dl. VI, blz. 559.

points taken on the curve coincide in a same point P, the limit of the first point is the centre C_p of the circle of curvature, that of the second point the centre S_p of the sphere of curvature, i. e. the centre of spherical curvature of R in P. If P describes the given curve R, then C_p and S_p describe twisted curves related to R, of which the latter is also the cuspidal edge of the developable enveloped by the normal planes of R; this locus of centres S_p of spherical curvature may be indicated by the symbol R_s .

From the wellknown theorem according to which the line of intersection c of two planes α, β, perpendicular to the intersecting lines a and b, is a normal to the plane γ of these lines a and b, ensues that reversely the osculating planes of R are also perpendicular to the corresponding tangents of R_s . These osculating planes of R however, not passing at the same time through the points of contact of the corresponding tangents of R_s , are not normal planes of R_s and so the relation between the curves R and R_s is generally not reciprocal. A wellknown striking example derived from transcendent twisted curves, where this reciprocity really exists, is the helix or curve formed by the thread of a screw; moreover for this curve the two loci of the points C_p and S_p coincide.

Let us go a step farther and suppose that $P_1, P_2, P_3, P_4, P_5, ..$ are successive points of a given curve R, which is contained in a four-dimensional space, but not in a three-dimensional one, which curve we therefore call a "wrung curve"; then besides the centres of the circle and sphere of curvature the centre H_p of the hypersphere of curvature appears, which is the limit of the hypersphere $P_1 P_2 P_3 P_4 P_5$, when the five determining points coincide in point P of the given curve. A third locus has then to be dealt with, and so we can extend these considerations to a space with any given number of dimensions.

In the following pages we wish to deduce the characteristics of the locus R_h of the centre of hyperspherical curvature of the highest rank in relation with the general rational wrung curve R_s^n of degree n, which *is* contained in a space with s dimensions but *not* in a space with s—1 dimensions.

2. "The row of characteristic numbers from class to degree of "the locus R_h of the centres of hyperspherical curve of the highest "rank belonging to the general rational wrung curve in R_s^n is

$$"3n-2, \quad 2(3n-3), \quad 3(3n-4), \; . \; . \; . \quad s(3n-s-1)."$$

To prove this we represent R_s^n by the equations

$$x_i = \frac{\alpha_i}{\nu}, \qquad (i = 1, 2, \ldots s) \quad \ldots\ldots \quad (1)$$

on rectangular axes, where the symbols $\alpha_1, \alpha_2, \ldots \alpha_s$ and ν indicate polynomia of degree n in a parameter t.

If the equations

$$\frac{\alpha_i}{\nu} = a_i + \frac{\beta_i}{\nu}, \qquad (i = 1, 2, \ldots s)$$

represent the result of the division of the s polynomia α_i by ν, where the s quantities a_i are independent of t and the s new polynomia β_i contain t in the degree $n-1$ at most, then it is clear that the transformation of the system of coordinates to parallel axes corresponding to the formulae

$$x_i = \xi_i + a_i, \qquad (i = 1, 2, \ldots s)$$

simplifies the original representation (1) of R_s^n into

$$\xi_i = \frac{\beta_i}{\nu}, \qquad (i = 1, 2, \ldots s) \quad \ldots\ldots \quad (2)$$

We repeat that this simplification consists in the fact that the s polynomia β_i ascend only to the degree $n-1$ in t.

If moreover β'_i and ν' represent the differential-coefficients of β_i and ν according to t, then

$$\nu \sum_{i=1}^{s} (\beta'_i \nu - \beta_i \nu') \xi_i = \sum_{i=1}^{s} (\beta'_i \nu - \beta_i \nu') \beta_i \quad \ldots \quad (3)$$

represents the normal space with $s-1$ dimensions of R_s^n in the point (2) with the value t of the parameter.

This equation is of degree $3n-2$ in t, which proves what was asserted. For the envelope of a space of $s-1$ dimensions, the equation of which contains a parameter to the degree k, has for characteristic numbers:

$$k, \quad 2(k-1), \quad 3(k-2), \ldots \quad s(k-s+1).$$

By means of the general theorem now proved we find from $n=2$ to $n=10$ the following table for the general rational twisted curve of minimum order:

$s=n=2$ . . . 4, 6,
$s=n=3$ . . . 7, 12, 15,
$s=n=4$ . . . 10, 18, 24, 28,
$s=n=5$ . . . 13, 24, 33, 40, 45,
$s=n=6$ . . . 16, 30, 42, 52, 60, 66,
$s=n=7$ . . . 19, 36, 51, 64, 75, 84, 91,
$s=n=8$ . . . 22, 42, 60, 76, 90, 102, 112, 120,
$s=n=9$ . . . 25, 48, 69, 88, 105, 120, 133, 144, 153,
$s=n=10$ . . . 28, 54, 78, 100, 120, 138, 154, 168, 180, 190.

The first line of this table says that the evolute of a general conic is a curve of class four and order six, the second that the locus R_s of the general skew cubic R_3^3 is a twisted curve of class seven, rank twelve and order fifteen, etc.

If as usual we consider the coefficients $u_1, u_2, u_3 \ldots . u_s$ of the equation $\Sigma u_i \xi_i = 0, (i=1, 2, \ldots s)$ as the tangential coordinates of the space with $s-1$ dimensions represented by that equation, we find from (3) for the normal space

$$u_i = -\frac{\nu(\beta_i' \nu - \beta_i \nu')}{\sum_{i=1}^{s} \beta_i (\beta_i' \nu - \beta_i \nu')}, \quad (i=1, 2, \ldots s), \quad . \quad (4)$$

which representation of R_h in space of s dimensions is dualistically opposite to that given for R_s^n. We write it in the abridged form:

$$u_i = \frac{\tau_i}{\tau_0}, \quad (i=1, 2, \ldots s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

3. The degree of the equation (3) or that of the forms τ of (5), all in t, can lower itself in particular circumstances. These, apparently of five kinds, can be reduced to the following two cases:

a). The equation $\nu=0$ has equal roots.

b). The equations $\beta_i = 0$, $(i=1, 2, \ldots s)$ have common equal roots.

We shall now consider the influence of each of those suppositions on the class of the locus R_h.

3^a. If $t = t_1$ is a k-fold root of $\nu = 0$, this value is at the same time a $k-1$-fold root of $\nu' = 0$ and each of the forms τ of (5), and so (3) too, is divisible by $(t-t_1)^{k-1}$. The curve R_h is then of class $3n - k - 1$.

By the substitution of $t - t_1 = \frac{1}{t'}$ the case of the k-fold root t_1 of $\nu = 0$ assumes an apparently different form. It transforms the equations (2) into

$$\xi_i = \frac{\gamma_i}{\mu}, \quad (i = 1, 2, \ldots s), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the s forms γ_i represent polynomia of degree n in t' without constant term, whilst μ contains t' to the degree $n-k$ only; so it leads to the case that $\mu = 0$, considered as an equation of degree n, possesses a k-fold root $t' = \infty$. Then the s forms $\tau_i, (i = 1, 2, \ldots s)$ of (5) become polynomia of degree $3n - 2k - 1$ in t', whilst τ_0 ascends to degree $3n - k - 1$ in t'. Then the corresponding equation (3) is also of degree $3n - k - 1$ in t' and so R_h remains of class $3n - k - 1$ as it should do.

In passing we draw attention to the fact that the degree of μ being lower than n it will be impossible to lower at the same time the degree of all the s polynomia γ_i by a transformation of coordinates to parallel axes, as this would include at the same time the possibility to lower the order of R_s^n.

The particular case treated here refers to the position of the points of R_s^n at infinity. If ν is divisible by $(t-t_1)^k$ the point at infinity of the curve belonging to t_1 will count k times among the n points of intersection of the curve with the space at infinity with $s-1$ dimensions containing all points at infinity of the space with s dimensions.

So we find for $s = n = 3$:

"The class of the locus R_s of a skew ellipse or a skew hyperbola is seven, whilst this number passes into six with the parabolic hyperbola and into five with the skew parabola."

What we find here agrees with the wellknown results for $s = n = 2$. Although through any point P of the plane of an ellipse or hyperbola four normals of this curve pass, we can fall from this point three normals only on the parabola, as the line connecting P with the point P_∞ at infinity of the parabola must be considered as an improper normal. Any point P of space is situated in seven normal planes of a skew ellipse or skew hyperbola, but only in six normal planes of a parabolic hyperbola and in five normal planes of a skew parabola, as the plane through the connecting line PP_∞ of P with the point of contact P_∞ at infinity with the plane V_∞ at infinity, perpendicular to the tangent p_∞ of the curve in P_∞, represents one improper normal plane for the last but one, and the coincidence of two improper normal planes for the last.

Of course the particularity treated here can appear more than once. If $\nu = 0$ contains the roots $t_1, t_2, \ldots \ldots t_p$ respectively $k_1, k_2, \ldots k_p$ times, where each of the p quantities k exceeds unity, the class of R_h is represented by $3n + p - 2 - \sum_{j=1}^{p} k_j$.

3[b]. If $t = t_1$ is a common k-fold root of the s equations $\beta_i = 0$, then this value is at the same time a common $k-1$-fold root of the s equations $\beta'_i = 0$ and the s forms of τ_i (5) are divisible by $(t-t_1)^{k-1}$, whilst τ_0 contains the factor $(t-t_1)^{2k-1}$; then again (3) is divisible by $(t-t_1)^{k-1}$ and the curve R_h is of class $3n - k - 1$.

By the substitution of $t-t_1 = \frac{1}{t'}$ the case treated here presents itself in an apparently different form. It leads to the equations (6), where now the s forms γ_i represent polynomia of degree $n - k + 1$ in t' without constant term and μ is a general form of degree n in t'. Regarded as equations of degree n in t', the s equations $\gamma_i = 0$ contain the common $k-1$-fold root $t' = \infty$ and the common simple root $t' = 0$. The s terms $\tau_i, (i = 1, 2, \ldots s)$ become polynomia of degree $3n - k - 1$ in t', whilst τ_0 ascends only to degree $3n-2k-1$ in t'. The corresponding equation (3) is then as above of degree $3n - k - 1$.

Apparently besides the cases treated up till now where the equation (3) lowers its degree, another entirely new case can be pointed out, namely that where the $s+1$ equations $\beta_i' = 0$, $\nu' = 0$ have a common k-fold root $t = t_1$. It is easy to see however that this apparent new case forms but a special case of what was treated above. If we start from the equations (1), because after all we shall directly have to transform the coordinates to parallel axes, then we have

$$\alpha'_i = (t - t_1)^k \varphi_i^{(n-k-1)}, \quad (i = 1, 2, \ldots s), \quad \nu' = (t - t_1)^k \varphi_0^{(n-k-1)},$$

when the s symbols $\varphi_i^{(n-k-1)}$ and $\varphi_0^{(n-k-1)}$ represent polynomia of degree $n-k-1$ in t. From this ensues by integration

$$\alpha_i = (t-t_1)^{k+1}\psi_i^{(n-k-1)}+b_i,\ (i=1,2,\ldots s),\ \nu=(t-t_1)^{k+1}\psi_0^{(n-k-1)}+b_0,$$

in which the quantities b_i and b_0 denote constants. So the transformation of coordinates to parallel axes characterized by the formulae

$$x_i = \xi_i + \frac{b_i}{b_0},\quad (i=1,\ 2,\ \ldots\ s)$$

finally gives

$$\xi_i = \frac{(t-t_1)^{k+1}\chi_i^{(n-k-1)}}{\nu},\quad (i=1,\ 2\ \ldots\ s),$$

by which we alight on the case that the s equations $\alpha_i=0$ belonging to (1) have a common $k+1$-fold root t_1, whilst ν moreover after being diminished by a constant quantity b_0 is divisible by $(t-t_1)^{k+1}$.

The particularity treated here appears only in the case when the curve R_s^n has singular points of a definite character. So the simplest case of a common double root t_1 of the s equations $\beta_i=0$ implies that the origin of each of the spaces of coordinates $\xi_i=0$ represents two of the n points of intersection with R_s^n, which with a view to the equality of the values of the parameter belonging to those points only then takes place when R_s^n shows a cusp in this point. We see at the same time that we have not generally enough enunciated the case sub 3^b). For from this appears that the particularity will come in as soon as R_s^n has a cusp anywhere. So the case sub 3^b) ought to run: "The equations $\alpha_i=0$, $(i=1,2,\ldots s)$ have common equal roots or a transformation of coordinates to parallel axes can call forth this particularity."

Of course the case may present itself that t_1 is a common equal root of the s equations $\beta_i=0$, but that the degree of multiplicity in relation to those equations differs. If t_1 is a k_1-fold root of $\beta_1=0$, a k_2 fold root of $\beta_2=0$, etc, then for k we must take the smallest number k_i.

If it happens p times that a transformation of coordinates to parallel axes implies the particularity indicated here, and if $k_1, k_2, \ldots k_p$ are the smallest numbers k for each of the correspond-

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ing values $t_1, t_2, \ldots t_p$ of t, then $3n+p-2 - \sum_{i=1}^{p} k_i$ will indicate the class of R_h.

4. In the preceding number we have dealt with the class of R_h only, without taking the other characteristic numbers into consideration. We now immediately add that the rule according to which the envelope of a space with $s-1$ dimensions, the equation of which contains a parameter to degree k, is characterized by the numbers

$$k, \quad 2(k-1), \quad 3(k-2), \ldots \ldots s(k-s+1)$$

in general needs some modifications as soon as one of the above-mentioned particular cases appears. In the very simplest case of the parabola we find e. g. for the characteristic numbers, class and order, of the evolute 3 and 3, but not 3 and 4 as might be expected for $k=3$. So in general in each of the particular cases treated here the numbers k, $2(k-1)$, $3(k-2)$, etc. must be treated as upper limits.

In a following paper we shall revert to this last point.

Physiology. — "*Lipolytic ferment in ascites-liquid of man*". (Remarks on the resorption of fat and on the lipolytic function of the blood). By Dr. H. J. HAMBURGER.

(Read January 27, 1900.)

In an essay published in the year 1880 CASH [1]) has contradicted the opinion that the emulsion of fat already takes place in the intestinal lumen. For he was never successful in separating an emulsion from the contents of the intestines by centrifugal force. And he did not much wonder at this: for the small intestine has an acid reaction, and with acid reaction no fat-emulsion can be produced.

This opinion of CASH does not seem quite correct to me. Giving to animals a meal containing much fat, HEIDENHAIN has found [2]), and so have I myself many a time, that a creamy surface can be taken off the mucosa of the small intestine, which, examined microscopically, contains small fat-globules. Nevertheless this layer

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1) Archiv f. Physiol. 1880. S. 323.

2) PFLÜGER's Archiv. 1888, supplement, S. 93.