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Mathematics. — "*On Orthogonal Comitants*". By Prof. JAN DE VRIES.

If we regard x_1 and x_2 as the coordinates of any point P with respect to the rectangular axes OX_1 and OX_2, the binary form

$$a_x^n \equiv (a_1 x_1 + a_2 x_2)^{(n)} \equiv a_{n,0}\, x_1^n + n\, a_{n-1,1}\, x_1^{n-1} x_2 + \binom{n}{2} a_{n-2,2}\, x_1^{n-2} x_2^2 + \\ \ldots\ldots + a_{0,n}\, x_2^n$$

is represented by n lines through O, containing the points for which the form a_x^n disappears.

If ξ_1 and ξ_2 are the coordinates of P with respect to the rectangular axes $O\Xi_1$ and $O\Xi_2$, between the quantities x_1, x_2 and ξ_1, ξ_2 exist relations of the form

$$x_1 = \lambda_{11}\, \xi_1 + \lambda_{12}\, \xi_2, \qquad \xi_1 = \lambda_{11}\, x_1 + \lambda_{21}\, x_2,$$

$$x_2 = \lambda_{21}\, \xi_1 + \lambda_{22}\, \xi_2, \qquad \xi_2 = \lambda_{12}\, x_1 + \lambda_{22}\, x_2.$$

If by these substitutions the form a_x^n is transformed into α_ξ^n, we have

$$a_x = a_1 x_1 + a_2 x_2 = (a_1 \lambda_{11} + a_2 \lambda_{21})\, \xi_1 + (a_1 \lambda_{12} + a_2 \lambda_{22})\, \xi_2,$$

so

$$\alpha_1 = \lambda_{11}\, a_1 + \lambda_{21}\, a_2,$$

$$\alpha_2 = \lambda_{12}\, a_1 + \lambda_{22}\, a_2.$$

This proves that the symbolical coefficients a_1, a_2 and α_1, α_2 are transformed into each other by the same substitution as the variables x_1, x_2 and ξ_1, ξ_2.

2. In order to obtain *comitants*, i.e. functions of a_1, a_2, x_1, x_2 that are invariant with respect to the indicated orthogonal transformations, we can start from the covariants

$$x_1^2 + x_2^2 \quad \text{and} \quad (x_1 - y_1)^2 + (x_2 - y_2)^2,$$

representing the square respectively of OP and of the mutual distance of two points P and Q, being therefore *absolute comitants.*

The second covariant can be replaced by

$$(x_1^2 + x_2^2) - 2(x_1 y_1 + x_2 y_2) + (y_1^2 + y_2^2) \equiv x_x - 2\,x_y + y_y\,,$$

whilst from the relation

$$(x_1 y_2 - x_2 y_1)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 y_1 + x_2 y_2)^2,$$

or

$$(xy)^2 = x_x\, y_y - x_y\, y_x$$

ensues that the covariant (xy) is related to the covariants x_x and $x_y \equiv y_x$.

Now from these three absolute comitants follows immediately the invariant character of the symbols

$$a_a\,, \quad a_b \quad \text{and} \quad (ab)\,.$$

According to the above these absolute invariant symbols are connected by the relation

$$(ab)^2 = a_a\, b_b - a_b^2\,.$$

So for the construction of orthogonal comitants we can dispose of the symbols

$$a_a\,,\quad a_b\,,\quad (ab)\,,\quad a_x\,,\quad (ax)\,,\quad x_x\,,\quad x_y\,,\quad (xy)\,.$$

Evidently *linear* invariants can be generated only by the symbol a_a and present themselves only in the study of forms of *even* degree.

Consequently the form a_x^{2n} possesses the linear absolute invariant [1])

$$a_a^n \equiv a_{2n,0} + n\, a_{2n-2,2} + \binom{n}{2} a_{2n-4,4} + \cdots + a_{0,2n}\,.$$

3. The quadratic form

$$a_x^2 \equiv a_{20}\, x_1^2 + 2\, a_{11}\, x_1 x_2 + a_{02}\, x_2^2$$

furnishes the invariants

$$a_a \quad \equiv a_{20} + a_{02}\,,$$

$$a_b^2 \quad \equiv a_{20}^2 + 2\, a_{11}^2 + a_{02}^2\,,$$

[1]) The existence of this invariant was proved by Mr. W. MANTEL by means of an infinitesimal transformation (Wiskundige Opgaven, Dl. VII, p. 148).

$$(ab)^2 \equiv 2\,(a_{20}\,a_{02} - a_{11}^2)\,,$$

united by the relation found above.

Whilst $(ab)^2 = 0$ points to the coinciding of the two lines indicated by $a_x^2 = 0$, a_a disappears when those lines are at right angles.

If c is the tangent of the angle formed by the two lines, their angular coefficients satisfy the relation

$$\left(\frac{m_1 - m_2}{1 + m_1\,m_2}\right)^2 = c^2\,,$$

or

$$4\,(a_{20}\,a_{02} - a_{11}^2) + c^2\,(a_{20} + a_{02})^2 = 0\,,$$

or

$$2\,(ab)^2 + c^2\,a_a\,b_b = 0\,,$$

or at last

$$(c^2 + 2)\;(ab)^2 + c^2\,a_b^2 = 0\,.$$

So the invariant a_b disappears when $c^2 = -\,2$.

4. By the interpretation of the substitution $x_y = 0$ or

$$y_1 : y_2 = x_2 : -\,x_1$$

follows immediately that the covariant

$$(ax)^2$$

disappears for two lines which are at right angles to the lines representing a_x^2.

The covariant

$$(ax)\,a_x$$

changes only its sign by the substitution $x_y = 0$.

So $(ax)\,a_x = 0$ represents two orthogonal lines.

Indeed the sum of the coefficients of x_1^2 and x_2^2 is equal to zero.

If $a_{11} = 0$, so that the lines of a_x^2 lie symmetrically with respect to the axes of coordinates, we have

$$(ax)\,a_x \equiv (a_{20} - a_{02})\,x_1\,x_2\,.$$

This proves that the covariant $(ax)\,a_x$ furnishes the bisectors of the angles of the lines $a_x^2 = 0$.

This result is confirmed by the following consideration:
By the equations

$$a_x\, a_y = 0 \qquad \text{and} \qquad x_y = 0$$

the pairs of lines are indicated lying respectively harmonically with the lines $a_x^2 = 0$ and with the isotropical lines $x_x = 0$.

And now these two involutions have the pair of rays in common of which the equation is obtained by eliminating y between

$$a_1\, a_x\, y_1 + a_2\, a_x\, y_2 = 0 \qquad \text{and} \qquad x_1\, y_1 + x_2\, y_2 = 0\,.$$

So the equation

$$(ax)\, a_x = 0$$

represents the orthogonal lines separating $a_x^2 = 0$ harmonically.

5. If we put

$$a_b\, a_x\, b_x \equiv g_x^2\,,$$

then with a view to the equivalence of the symbols a and b, we have

$$g_x\, g_y = a_b\, a_x\, b_y\,,$$

and

$$g_x\,(gx) = a_b\, a_x\,(bx)\,.$$

But from the identical relation

$$a_a\, b_x - b_a\, a_x = (ab)\,(ax)$$

follows

$$(gx)\, g_x = a_a\,(bx)\, b_x - (ab)\,(ax)\,(bx)\,.$$

The second term of the right member disappearing identically and $(bx)\, b_x = 0$ representing the bisectors of the angles of the lines $b_x^2 = 0$, the covariant

$$a_b\ a_x\ b_x$$

furnishes two lines having in common with the lines of a_x^2 the axes of symmetry.

At the same time it is evident that the form $a_b\, a_x\,(bx)$ does not give a new covariant.

It is clear that $a_b\,(a\iota)\,(b\iota)$ represents the lines at right angles to $a_b\,a_x\,b_x = 0$.

6. To two quadratic forms a_x^2 and f_x^2 belong the simultaneous invariants

$$(af)^2\,, \qquad a_f^2 \equiv f_a^2\,, \qquad (af)\,a_f\,.$$

As is known the first disappears when the two pairs of lines separate each other harmonically.

Under the condition $(af)\,a_f = 0$ the lines determined by $(af)\,a_x\,f_x = 0$ are perpendicular to one another.

These right lines being the double rays of the involution $a_x^2 + \lambda f_x^2 = 0$, the equation $(af)\,a_f = 0$ indicates that the pair of lines $a_x^2 = 0$ and $f_x^2 = 0$ have common axes of symmetry.

This is confirmed by the following consideration. We have

$$(af)\,a_f = (a_{20} - a_{02})\,f_{11} - a_{11}\,(f_{20} - f_{02})\,.$$

If $f_{11} = 0$, the invariant disappears when at the same time $a_{11} = 0$ or $f_{20} = f_{02}$, i. e. when the two pairs of rays have common bisectors or when one pair of rays consists of the isotropical lines.

From the expression found above for the tangent of the angles of a pair of lines follows readily that the invariant

$$(ab)^2 f_f\, g_g - (fg)^2\, a_a\, b_b$$

disappears, when these pairs of lines can be brought to coincidence by the rotation of one of them.

7. When the equations

$$a_x^2 \equiv a_{20}\,x_1^2 + 2\,a_{11}\,x_1\,x_2 + a_{02}\,x_2^2 = 0\,,$$

$$(fx)^2 \equiv f_{02}\,x_1^2 - 2\,f_{11}\,x_1\,x_2 + f_{20}\,x_2^2 = 0$$

have a root $x_1 : x_2$ in common, one of the lines $a_x^2 = 0$ is at right angles to a line of $f_x^2 = 0$. So the resultant of these equations must furnish a simultaneous invariant.

By elimination of $x_1 : x_2$ we find

$$(a_{20} f_{20} - a_{02} f_{02})^2 + 4 (a_{20} f_{11} + a_{11} f_{02}) (a_{02} f_{11} + a_{11} f_{20}) = 0 .$$

By a simple computation this expression is reduced to the symbolic form

$$a_f^2 b_g^2 - 2 (ab) (fg) a_f b_g = 0 ,$$

where $a_x^2 \equiv b_x^2$ and $f_x^2 \equiv g_x^2$.

If in the preceding equations we put $a_{20} = 0$, then we have $f_{02} = 0$ or $a_{02}^2 f_{02} + 4 a_{02} a_{11} f_{11} + 4 a_{11}^2 f_{20} = 0$. In the former case a line of a_x^2 is at right angles to a line of f_x^2. In the latter case the substitution $a_{02} = 2 i a_{11}$ furnishes the condition

$$- f_{02} + 2 i f_{11} + f_{20} = 0 ,$$

from which ensues that one of the two isotropical lines belongs to each of the pairs; then again a line of a_x^2 is at right angles to a line of f_x^2.

The consideration of the orthogonal pair of rays of the involution

$$a_x^2 + \lambda f_x^2 = 0$$

leads to a simultaneous covariant.

This pair is indicated by

$$(a_1^2 + \lambda f_1^2) + (a_2^2 + \lambda f_2^2) = 0 ,$$

or by

$$a_a + \lambda f_f = 0 ,$$

so by

$$f_f a_x^2 - a_a f_x^2 = 0 .$$

8. It is a matter of course, that to the cubic form

$$a_x^3 \equiv a_{30} x_1^3 + 3 a_{21} x_1^2 x_2 + 3 a_{12} x_1 x_2^2 + a_{03} x_2^3$$

belong only invariants with an even number of symbols, i. e. of even degree in the coefficients.

Setting aside the forms $(ab)\, a_b^2$ and $(ab)^3$ which disappear identically, we have the invariants

$$(ab)^2\, a_b = 2\,(a_{30}\, a_{12} - a_{21}^2 - a_{12}^2 + a_{03}\, a_{21})\,,$$

$$a_b^3 = a_{30}^2 + 3\, a_{21}^2 + 3\, a_{12}^2 + a_{03}^2\,.$$

From the identity $(ab)^2 + a_b^2 = a_a\, b_b$ evidently follows

$$a_a\, a_b\, b_b \equiv (ab)^2\, a_b + a_b^3\,.$$

For $a_{30} = 0$ and $a_{03} = 0$ we have

$$a_b^3 = 3\,(a_{21}^2 + a_{12}^2) \text{ and } (ab)^2\, a_b = -\,2\,(a_{21}^2 + a_{12}^2)\,,$$

so

$$2\, a_b^3 + 3\,(a\,b)^2\, a_b \equiv 2\,(a_{30}^2 + 3\, a_{30}\, a_{12} + 3\, a_{03}\, a_{21} + a_{03}^2) = 0.$$

Reciprocally the disappearing of this invariant indicates that two lines of $a_x^3 = 0$ are at right angles to each other. For, if by a rotation of the axes of coordinates a_x^3 is transformed into

$$3\, \alpha_{21}\, \xi_1^2\, \xi_2 + 3\, \alpha_{12}\, \xi_1\, \xi_2^2 + \xi_2^3\,,$$

which implies that one of the lines is represented by $\xi_2 = 0$, then the angular coefficients of the remaining lines are connected by the relation $m_2\, m_3 = 3\, \alpha_{21}$. The above named invariant being transformed into $3\, \alpha_{21} + 1$, its disappearing produces the relation $m_2\, m_3 + 1 = 0$, by which two perpendicular lines are indicated.

9. The comitant

$$a_y\, a_x^2 = 0$$

determines the polar of a_x^3 with respect to the line $y_1 : y_2 = x_1 : x_2$, or, what comes to the same thing, the double lines of the cubic involution of which $(x\,y) = 0$ is a threefold ray and $a_x^3 = 0$ forms a group.

For the double rays of the involution

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$$(x\,y)^3 + \lambda\, a_x^3 = 0$$

are determined by

$$\begin{vmatrix} (x\,y)^2\, y_2 & , & a_x^2\, a_1 \\ -\,(x\,y)^2\, y_1 & , & a_x^2\, a_2 \end{vmatrix} = 0,$$

or by $a_y\, a_x^2 = 0$.

In connection with this consideration the covariant of HESSE

$$(a\,b)^2\, a_x\, b_x$$

furnishes two lines forming the threefold elements of a cubic involution of which $a_x^3 = 0$ is a group.

The lines of HESSE are orthogonal, when the invariant $(a\,b)^2\, a_b$ is equal to zero.

The lines $a_y\, a_x^2 = 0$ are orthogonal when the covariant $a_a\, a_y$ disappears, i. e. when we have

$$y_1 : y_2 = a_2\, a_a : -\, a_1\, a_a.$$

By substitution into $b_y\, b_x^2 = 0$ we find that the pair of rays in question is indicated by

$$(a\,b)\, a_a\, b_x^2 = 0.$$

The lines of HESSE are the double elements of the involution

$$(a\,b)^2\, a_x\, b_y = 0.$$

If $y_1 : y_2$ is replaced by $c_2\, c_c : -\, c_1\, c_c$, it is evident that the covariant

$$(a\,b)^2\,(b\,c)\, c_c\, a_x$$

determines the ray conjugate in this involution to the ray $a_a\, a_y = 0$.

Evidently the orthogonal pair of rays of this involution is indicated by

$$(a\,b)^2\, a_x\,(b\,x) = 0.$$

So this must correspond with $(a\,b)\,a_a\,b_x^2 = 0$, which indicates according to the above the same pair of rays. By applying the identity $(a\,b)\,(a\,x) = a_a\,b_x - b_a\,a_x$, we find $(a\,b)^2 b_x\,(a\,x) = (a\,b)\,a_a b_x^2 - (a\,b)\,b_a\,a_x\,b_x$, where the third covariant disappears identically.

10. The biquadratic form

$$a_x^4 \equiv a_{40}\,x_1^4 + 4\,a_{31}\,x_1^3\,x_2 + 6\,a_{22}\,x_1^2\,x_2^2 + 4\,a_{13}\,x_1\,x_2^3 + a_{04}\,x_2^4$$

has, beside the above mentioned invariant

$$h \equiv a_a^2 \equiv a_{40} + 2\,a_{22} + a_{04}$$

and the well known invariants

$$i \equiv (a\,b)^4 \quad \text{and} \quad j \equiv (a\,b)^2\,(a\,c)^2\,(b\,c)^2,$$

the quadratic invariants

$$m \equiv a_b^4 \equiv a_{40}^2 + 4\,a_{31}^2 + 6\,a_{22}^2 + 4\,a_{13}^2 + a_{04}^2,$$

$$l \equiv (a\,b)^2\,a_b^2 \equiv 2\,a_{22}\,(a_{40} - 2\,a_{22} + a_{04}) - 2\,(a_{31} - a_{13})^2\,.$$

In consequence of the identity $(a\,b)^2 + a_b^2 = a_a\,b_b$ we have

$$(a\,b)^4 + 2\,(a\,b)^2\,a_b^2 + a_b^4 = a_a^2\,b_b^2\,,$$

so that we have also

$$i + 2\,l + m = h^2.$$

If we put $a_{40} = 0$ and $a_{04} = 0$, then

$$h = 2\,a_{22},$$

$$i = 2\,(3\,a_{22}^2 - 4\,a_{31}\,a_{13}),$$

$$j = 6\,(2\,a_{31}\,a_{22}\,a_{13} - a_{22}^3),$$

so

$$3\,h^3 = 6\,i\,h + 8\,j.$$

Consequently the invariant

$$8\,(ab)^2\,(ac)^2\,(bc)^2 + 6\,(ab)^4\,c_c^2 - 3\,a_a^2\;b_b^2\;c_c^2$$

disappears when the four lines $a_x^4 = 0$ contain two pairs at right angles.

11. Ternary forms can be represented by cones the top of which is situated in the origin of three axes of coordinates perpendicular to one another.

Here too it is evident that by a rotation of the axes of coordinates the symbolical coefficients a_k of the form

$$a_x^n \equiv (a_1\,x_1 + a_2\,x_2 + a_3\,x_3)^{(n)}$$

undergo the same substitution as the coordinates.

So the comitants

$$x_1^2 + x_2^2 + x_3^2\,, \qquad x_1\,y_1 + x_2\,y_2 + x_3\,y_3\,,$$

$$(x_1\,y_2 - x_2\,y_1)^2 + (x_2\,y_3 - x_3\,y_2)^2 + (x_3\,y_1 - x_1\,y_3)^2,$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

furnish the invariant symbols

$$a_a\,, \qquad a_b\,, \qquad (a_1\,b_2)^2 + (a_2\,b_3)^2 + (a_3\,b_1)^2$$

and

$$(abc)\,.$$

For quadratic cones we immediately find the orthogonal invariants

$a_a \equiv a_{11} + a_{22} + a_{33}\,,\quad \Sigma\,(a_{11}\,a_{22} - a_{12}^2)$ and $(abc)^2 = \Sigma \pm a_{11}\,a_{22}\,a_{33}\,.$