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whilst the differential equation for v reduces to

$$\frac{\partial v}{\partial z} = \frac{\partial \varrho}{\partial x}.$$

Let σ be an arbitrary function of x, y, z. We shall now find

$$\mu = \frac{1}{\frac{\partial \sigma}{\partial z}} \left[p \left(\frac{\partial^2 \sigma}{\partial y\, \partial z} + q \frac{\partial^2 \rho}{\partial z^2} \right) + \frac{\partial^2 \sigma}{\partial x\, \partial y} + q \frac{\partial^2 \sigma}{\partial x\, \partial z} \right].$$

The differential equation $s + \mu = 0$, for which may be written

$$\frac{d^2 \sigma}{dx\, dy} = 0,$$

possesses as intermediate integrals

$$\frac{d\sigma}{dx} = \frac{\partial \sigma}{\partial x} + p \frac{\partial \sigma}{\partial z} = f(x),$$

$$\frac{d\sigma}{dy} = \frac{\partial \sigma}{\partial y} + q \frac{\partial \sigma}{\partial z} = f(y),$$

where f denotes an arbitrary function.

These results differ in form only from those formerly communicated sub V.

Mathematics. — "*On the locus of the centre of hyperspherical curvature for the normal curve of n-dimensional space*". By Prof. P. H. Schoute.

At the close of the preceding paper we have pointed out that the characteristic numbers of the locus of the centre of hyperspherical curvature are lowered if some of the points of the given rational curve lying at infinity coincide. At present we wish to trace for a special case the amount of those lower numbers, viz. for the case where the given curve is the "normal curve" N_n^n of the n-dimensional space S_n, in which it is situated. It is known that this curve is represented on rectangular coordinates by the equations

$$x_i = t^i, \quad (i = 1, 2, \ldots n) \;, \quad \ldots \ldots \quad (1)$$

where t is again the parametervalue of the "point t" of the curve.

The quintic ν of the preceding paper being unity here, $\nu = 0$ considered as an equation of degree n has here n infinite roots, from which ensues that the n points at infinity of the curve coincide in a single point, the point at infinity of the x_n-axis. As an introduction to the general case of an arbitrary n, let us first however consider the case $n = 3$ of the skew parabola.

1. If to avoid indices we write for the rectangular coordinates of a point of S_3 as is customary x, y, z instead of x_1, x_2, x_3, the skew parabola is represented by

$$x = t, \quad y = t^2, \quad z = t^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The equation of the normal plane in the point t is

$$x - t + 2t(y - t^2) + 3t^2(z - t^3) = 0,$$

or classified according to t

$$3t^5 + 2t^3 - 3zt^2 + (1 - 2y)t - x = 0 \quad . \quad . \quad . \qquad (3)$$

This equation being of degree 5 in t, five normal planes of the skew parabola pass through any given point and so the locus R_s is of order five, as was formerly found.

The equation of the developable enveloped by the series of normal planes is found by eliminating t out of (3) and its differential coefficient according to t. This is immediately reduced to the elimination of t between the two cubic equations

$$\left.\begin{array}{l} 4t^3 \qquad\qquad - 9zt^2 + 4(1-2y)t \qquad\quad - 5x = 0 \\ 135zt^3 + 12(10y-7)t^2 + 3(25x-8)t + 4(1-2y) = 0 \end{array}\right\},$$

by which is found by means of the wellknown method of elimination

$$\begin{vmatrix} 4, & -9z, & 4-8y, & -5x, & 0, & 0 \\ 0, & 4, & -9z, & 4-8y, & -5x, & 0 \\ 0, & 0, & 4, & -9z, & 4-8y, & -5x \\ 135z, & 120y-84, & 75x-24, & 4-8y, & 0, & 0 \\ 0, & 135z, & 120y-84, & 75x-24, & 4-8y, & 0 \\ 0, & 0, & 135z, & 120y-84, & 75x-24, & 4-8y \end{vmatrix} = 0.$$

So the developable referred to is of degree six; so six is the rank of R_s.

By solving x, y, z out of (3) and its first and second differential coefficients according to t we find

$$\left.\begin{aligned} x &= 2t^3(9t^2+1) \\ 1-2y &= 3t^2(15t^2+2) \\ z &= 2t(5t^2+1) \end{aligned}\right\} , \quad \ldots\ldots \quad (4)$$

from which ensues that the curve R_s is of degree five. So, instead of 5, $2(5-1)$, $3(5-2)$ or $5, 8, 9$, the characteristic numbers of R_s are $5, 6, 5$.

In passing we can remark here, that the normal plane

$$2(3t^2x-3ty+z)=108t^7+147t^5+38t^3+t \quad \ldots \quad (5)$$

of the curve R_s in the point t is parallel, as it should be, to the plane of curvature

$$t^3-3t^2x+3ty-z=0$$

of the skew parabola in the point t. The equation (5) being of degree seven in t, the locus R'_s, belonging to R_s as original curve, is of class seven. This agrees with the general result obtained in the preceding paper. For the number $3n-2$, here 13, must be diminished by four on account of the particularity sub ^{a}) and by two on account of the particularity sub ^{b}). For, ν being a constant, the quintic equation $\nu=0$ has five equal infinite roots; moreover the three equations $\alpha'_1=0$, $\alpha'_2=0$, $\alpha'_3=0$ have the factor $15t^2+1$ in common, in connection with which the curve R'_s proves to contain two conjugate complex cusps.

2. The method followed here for $n=3$ not being so easy to apply to the space S_n, we shall try to find another way, where that drawback does not present itself. To do so we must recall in mind the proof of the theorem formerly used, according to which the envelope of a space with $n-1$ dimensions, of which the equation, linear in the coordinates x_i $(i=1, 2, \ldots n)$, contains a parameter t to degree k, has the characteristic numbers

$$k, \quad 2(k-1), \quad 3(k-2), \ldots\ldots (n-1)(k-n+2),$$

where it is taken for granted that $k>n-2$, as otherwise the last envelope contains *either* a morefold infinite number of points and is then not a curve, *or* — in case it really consists of a singly infinite number of points — it is situated in a space S_{n-1}. Here k is always $2n-1$.

The indicated proof can be given by means of the two following considerations:

a). The system of $s+1$ equations consisting of the equation of degree k

$$f(t) \equiv a_0 t^k + a_1 t^{k-1} + \ldots . a_{k-1} t + a_k = 0$$

and its first, second.... s^{th} differential coefficients according to t may be replaced by a system of $s+1$ equations of degree $k-s$ in t, all admitting coefficients that are linear forms of the coefficients of $f(t) = 0$.

b). The degree of the locus represented by $s+1$ equations of degree $k-s$ in t, of which the coefficients are linear forms in the coordinates $x_i\,(i = 1, 2, \ldots n)$, is obtained by adding to the system $n-s$ entirely arbitrary equations linear in the coordinates and by eliminating the n coordinates between the so formed system of $n+1$ equations of which $n-s$ do not contain t. The degree of the resulting equation in t is the order of the locus we were in search of.

The proof of these two lemmae is very simple. The first is but an extension of a wellknown theorem of EULER. If we transform the equation $f(t) = 0$ by the substitution $t = \frac{u}{v}$ into the homogeneous form $\varphi(u, v) = 0$, the $s+1$ indicated equations are

$$\frac{\partial^s \varphi}{\partial u^s} = 0, \quad \frac{\partial^s \varphi}{\partial u^{s-1}\, \partial v} = 0 \ldots , \frac{\partial^s \varphi}{\partial v^s}.$$

And by following the method pointed out in the second lemma we find the number of points common to the locus of $n-s$ dimensions, determined by the $s+1$ equations of degree $k-s$ and the space S_s, being the intersection of any system of $n-s$ spaces S_{n-1}.

If the condition is written down, that the eliminant of the system of $n+1$ equations, linear in the n coordinates, disappears, we obtain an equation of degree $(s+1)(k-s)$ in t, which proves the theorem.

3. It goes without saying that the lowering, which the characteristic numbers of the locus R_s belonging to the skew parabola undergo, is closely connected with the particular structure of the equation. First, this equation is not complete, for t^4 is lacking; secondly, not all existing terms contain the three coordinates x, y, z

in their coefficients. We shall first point out, that the latter peculiarity explains the lowering appearing here even then, if we neglect to avail ourselves of the simplification indicated in the lemma a); we shall then show that the first particularity has no effect here.

By substituting in the eliminant of the system for each element the number indicating its degree in t and by representing the places made vacant by differentiation by the symbol †, then in the three cases $s = 0, 1, 2$, appearing in the skew parabola, we have — independent of the lacking of t^4 in (3) — to deal with the three symbolic equations

$$\begin{vmatrix} 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & 1 & 2 & 5 \\ \dagger & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & 1 & 2 & 5 \\ \dagger & 0 & 1 & 4 \\ \dagger & \dagger & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0,$$

which really show that the corresponding equations in t are respectively of degree 5, 6, 5.

By substituting furthermore in the eliminant for each element the term of the highest degree in t, we then find omitting the first case, clear enough in itself,

$$\begin{vmatrix} 1 & t & t^2 & t^5 \\ \dagger & 1 & 2t & 5t^4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & t & t^2 & t^5 \\ \dagger & 1 & 2t & 5t^4 \\ \dagger & \dagger & 2 & 20t^3 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix} = 0$$

and now, taking the arbitrariness of the coefficients a, b of the equations of the planes S_2 into consideration, it is clear that the terms of the highest degree

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} t^2 & t^5 \\ 2t & 5t^4 \end{vmatrix} = 3\,(a_1 b_2)\, t^6, \; - a_1 \begin{vmatrix} t & t^2 & t^5 \\ 1 & 2t & 5t^4 \\ \dagger & 2 & 20t^3 \end{vmatrix} = -\,12\, a_1\, t^5$$

and the constant terms

$$(a_3 b_4), \quad 2\, a_4$$

of these equations cannot be expelled by applying the method of the first lemma or by making use of the lacking of t^4 in (3), by which equations $f(t)=0$ etc. of still lower degree are obtained. For, these operations correspond with the diminishing of the elements of a row of the determinant indicated above by the corresponding elements of another now multiplied by a form in t, and by this method of transformation, much in use with determinants, the degree of the determinant in t cannot be lowered. So it is only apparently that by applying the first lemma the degree of the general eliminant is lowered from

$$k+(k-1)+\ldots+(k-s)=\frac{(2k-s)(s+1)}{2} \text{ to } \frac{(2k-2s)(s+1)}{2};$$

in reality the eliminant of the equations

$$f(t)=0,\ \frac{\partial f}{\partial t}=0, \ldots \frac{\partial^s f}{\partial t^s}=0$$

is already of degree $(s+1)(k-s)$, although judging by the form it seems to be of a higher degree. On the other hand in the case of the skew parabola

$$\begin{vmatrix} 0 & 1 & 2 & 5 \\ \dagger & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}=0 \text{ passes into } \begin{vmatrix} 0 & 1 & 2 & 3 \\ \dagger & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}=0 \text{ and } \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}=0,$$

if in succession we make use of the method of the first lemma or of the two cubic equations used in the direct solution; so the determinant remains of degree six in t.

4. We are now able to treat the general case completely, where n and $s<n$ are arbitrary and k is equal to $2n-1$. If as is customary we represent the analytical faculty

$$p(p+r)(p+2r)\ldots\{p+(q-1)r\}$$

by $p^{q|r}$ the equation under investigation appears in the form

$$\begin{vmatrix} 1 & t & t^2 & \dots & t^{n-s-1} & t^{n-s} & \dots & t^{n-1} & t^{2n-1} \\ \dagger & 1 & 2t & \dots & (n-s-1)t^{n-s-2} & (n-s)\,t^{n-s-1} & \dots & (n-1)\,t^{n-2} & (2n-1)\,t^{2n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & (n-s)^{s|1}t^{n-s-1} & (2n-s)^{s|1}t^{2n-s-1} \\ a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n-s} & a_{1,n-s+1} & \dots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n-s} & a_{2,n-s+1} & \dots & a_{2,n} & a_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-s,1} & a_{n-s,2} & a_{n-s,3} & \dots & a_{n-s,n-s} & a_{n-s,n-s+1} & \dots & a_{n-s,n} & a_{n-s,n+1} \end{vmatrix} = 0.$$

By multiplying the second row by t, the third by t^2 etc. and the $s+1^{st}$ by t^s, the first $s+1$ elements of each column assume the same power of t. From this ensues that

$$(n-s) + (n-s+1) + \dots + (n-1) + 2n-1$$

diminished by

$$1 + 2 + \dots \quad + s$$

or $2n-1+s(n-s-1)$ indicates the degree of the equation, if the terms of the highest degree and the constant term do not disappear. The constant term is the product of the numbers $1, 2, 6, 24, \dots$, and a determinant of coefficients $a_{i,k}$; so this does not disappear. And taken together the terms of the highest degree $2n-1+s(n-s-1)$ have as coefficient the product of a determinant of quantities $a_{i,k}$ and

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ n-s & n-s+1 & \dots & n-1 & 2n-1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & (n-s)^{s|1} & (2n-s)^{s|1} \end{vmatrix},$$

which is reducible to

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ n-s & n-s+1 & \dots & n-1 & 2n-1 \\ (n-s)^2 & (n-s+1)^2 & \dots & (n-1)^2 & (2n-1)^2 \\ \dots & \dots & \dots & \dots & \dots \\ (n-s)^s & (n-s+1)^s & & (n-1)^s & (2n-1)^s \end{vmatrix}$$

and this differs from zero, it being the product of all possible differences of the $s+1$ numbers $n-s$, $n-s+1$, ... $n-1$, $2n-1$ and no equal ones appearing among them.

According to the final result obtained in this way the characteristic numbers of the locus of the centre of hyperspherical curvature R_h for the normal curve N_n^n are respectively

$$2n-1,\ 3n-3,\ 4n-7,\ 5n-13,\ 6n-21, \dots . \ 2n-1,$$

from which ensues that they do not change if taken in reversed order. In particular we find for

$n = 2$ 3, 3
$n = 3$ 5, 6, 5
$n = 4$ 7, 9, 9, 7
$n = 5$ 9, 12, 13, 12, 9
$n = 6$ 11, 15, 17, 17, 15, 11
$n = 7$ 13, 18, 21, 22, 21, 18, 13
$n = 8$ 15, 21, 25, 27, 27, 25, 21, 15
$n = 9$ 17, 24, 29, 32, 33, 32, 29, 24, 17
$n = 10$ .... 19, 27, 33, 37, 39, 39, 37, 33, 27, 19.

With this the table inserted in the preceding paper referring to the general rational skew curve of minimum degree can be compared.

Physics. — "*Equations in which functions occur for different values of the independent variable*". By J. D. VAN DER WAALS JR. (Communicated by Prof. J. D. VAN DER WAALS.)

§ 1. Let us imagine an electric vibrator at a distance r from a reflecting plane. If we wish to construe the equation of motion of that vibrator at the moment t we shall have to take into consideration that forces act on the vibrator, which it has given out itself and which have then been reflected by the plane. These forces are determined