$$x_i = t^i, \quad (i = 1, 2, \ldots \ldots n),$$

the following theorems are proved in an entirely similar way:

"The hyperspheres H_{n-1} with $n-1$ dimensions determined by "groups of $n+1$ points of the curve N^n_n conormal with $n-2$ given "points $t_1, t_2, \ldots \ldots t_{n-2}$ of that curve, intersect the curve still in "$n-1$ fixed points $s_1, s_2, \ldots s_{n-1}$ and form therefore a net, of "which the hypersphere H_{n-3} determined by those $n-1$ points s is "the base. And if the system of the $n-2$ given points describes "the curve N^n_n, the groups of $n-1$ points s determine on N^n_n an invo-"lution of degree $n-1$ with $n-2$ dimensions."

"If the point P describes the plane π common to the normal "spaces of $n-2$ given points $t_1, t_2, \ldots \ldots t_{n-2}$, the centre M of the "corresponding hypersphere of JOACHIMSTHAL describes the plane μ, "which is the locus of the points at equal distance from the "$n-1$ points s depending in the indicated way on the $n-2$ given "points t; then P and M describe in the planes π and μ affinely "related point-fields (P) and (M)."

"Between the spacial systems (P) and (M) with n dimensions "exists a correspondence $(2n-1, 2n-1)$."

"The cyclographic representation of all the hyperspheres of "JOACHIMSTHAL demands the given space S_n to be supposed to be "part of a S_{n+1}; it leads to a curved space of order $2\,(2n-1)$ with "n dimensions in this S_{n+1} as locus of the pairs of images, etc."

We believe we can suffice with these general indications; we only wish still to observe that the coefficients of the equation determining the $n-1$ fixed points are connected in a simple way with the symmetrical functions $\sum_{n-2,\,k} t$ of the $n-2$ points t taken arbitrarily.

Mathematics. — "*Approximation formulae concerning the prime numbers not exceeding a given limit*". By Prof. J. C. KLUYVER.

RIEMANN's method for determining the total number of the prime numbers p less than a given number c is equally serviceable, when it is required to evaluate other arithmetic expressions involving these prime numbers, for example the sum $\sum_{p<c} p^{-s}$ of their $(-s)^{\text{th}}$ powers, or the least common multiple $M(c)$ of all integers less than c. The different results, thus deduced, constantly contain a set of terms depending on the complex zeros μ of the RIEMANN ζ-function. The most important of these terms is always one and the same disconti-

nuous functions of c; the remaining ones are continuous and of less consequence.

A direct evaluation of the discontinuous term is not to be thought of; if however we suppose given beforehand the number of the prime numbers less than c, we can eliminate the discontinuous term and we are enabled, as will be shown subsequently, to arrive at rather close approximations of other more or less symmetric functions of these prime numbers.

In order to obtain the formulae we have in view, we must apply RIEMANN's method to the discontinuous integral

$$G_s(c) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{c^z}{z} \log \zeta(z+s)\, dz,$$

the path of integration being a straight line parallel to the imaginary axis and on the positive side of it.

By EULER's equation we have, supposing $z+s>1$,

$$\log \zeta(z+s) = \sum_{n=1}^{n=\infty} \frac{1}{n} \Sigma\, p^{-z-s}$$

and by inserting this value in the integral we find

$$G_s(c) = \sum_{n=1}^{n=\infty} \frac{1}{n} \sum_{p^n<c} p^{-ns}.$$

On the other hand we may express $\zeta(z+s)$ as an infinite product, that is we may write

$$\log \zeta(z+s) = -\log 2 - \log(z+s-1) + \frac{z+s}{2}(C+\log \pi) +$$

$$+ \sum_{\mu} \log\left(1-\frac{z+s}{\mu}\right) + \sum_{n=1}^{n=\infty} \left\{ \log\left(1+\frac{z+s}{2n}\right) - \frac{z+s}{2n} \right\},$$

or by subtracting at both sides $\log \zeta(s)$,

$$\log \zeta(z+s) = \log \zeta(s) - \log\left(1-\frac{z}{1-s}\right) + \frac{z}{2}(C+\log \pi) +$$

$$+ \sum_{\mu} \log\left(1-\frac{z}{\mu-s}\right) + \sum_{n=1}^{n=\infty} \left\{ \log\left(1+\frac{z}{2n+s}\right) - \frac{z}{2n} \right\},$$

where in the series

$$\sum_{\mu} log\left(1-\frac{z}{\mu-s}\right)$$

the terms should be arranged according to the ascending moduli of the μ's. Using this expansion of $log\,\zeta(z+s)$ in the integral $G_s(c)$ and adhering strictly to the reasoning adopted by RIEMANN for the integral $G_0(c)$, we obtain

$$G_s(c)=log\,|\,\zeta(s)|+Li\,(c^{-s+1})+\int_c^{\infty}\frac{dt}{t^{1+s}(t^2-1)\,log\,t}-\sum_{\mu} Li\,(c^{\mu-s})\;.$$

It is desirable to give a somewhat different form to the series of integrallogarithms arising here. Let us assume the zeros μ to be of the form $\frac{1}{2}\pm i\beta$, then, by pairing off conjugate complexes, we have

$$\sum_{\mu} Li\,(c^{\mu-s})=\sum_{\mu}\left[Li\,(c^{\frac{1}{2}-s+i\beta})+Li\,(c^{\frac{1}{2}-s-i\beta})\right]=$$

$$=\sum_{\beta}\left[\int_{-\infty+i\beta}^{\frac{1}{2}-s+i\beta}\frac{c^x}{x}\,dx+\int_{-\infty-i\beta}^{\frac{1}{2}-s-i\beta}\frac{c^x}{x}\,dx\right]=$$

$$=2\sum_{\beta}\left[cos\,(\beta\,log\,c)\int_{-\infty}^{\frac{1}{2}-s}\frac{c^x\,x\,dx}{x^2+\beta^2}+\beta\,sin\,(\beta\,log\,c)\int_{-\infty}^{\frac{1}{2}-s}\frac{c^x\,dx}{x^2+\beta^2}\right]=$$

$$=2\sum_{\beta}\left[\frac{sin\,(\beta\,log\,c)}{\beta}\int_{-\infty}^{\frac{1}{2}-s}c^x\,dx+\frac{cos\,(\beta\,log\,c)}{\beta^2}\int_{-\infty}^{\frac{1}{2}-s}c^x\,x\,dx-\right.$$

$$\left.-\frac{1}{\beta^2}\int_{-\infty}^{\frac{1}{2}-s}\frac{c^x\,x^2\,dx}{\sqrt{x^2+\beta^2}}\cdot sin\left(\beta\,log\,c+sin^{-1}\frac{x}{\sqrt{x^2+\beta^2}}\right)\right]=$$

$$=\frac{2\,c^{\frac{1}{2}-s}}{log\,c}\left[\sum_{\beta}\frac{sin\,(\beta\,log\,c)}{\beta}+\left(\tfrac{1}{2}-s-\frac{1}{log\,c}\right)\sum_{\beta}\frac{cos\,(\beta\,log\,c)}{\beta^2}+\varrho\sum_{\beta}\frac{1}{\beta^3}\right],$$

ϱ denoting a quantity the absolute value of which is less than

$$\left|\;(\tfrac{1}{2}-s)^2-\frac{2}{log\,c}(\tfrac{1}{2}-s)+\frac{2}{(log\,c)^2}\;\right|.$$

Dealing similarly with the integral

$$\int_c^\infty \frac{dt}{t^{1+s}\,(t^2-1)\log t},$$

it appears that we may replace it by

$$\frac{\theta}{(2+s)\,(c^2-1)^{1+\frac{1}{2}s}\log c},$$

where θ is positive and less than unity.

Hence the preceding equation for $G_s(c)$ can be written

$$G_s(c) = \log|\zeta(s)| + Li(c^{-s+1}) + \frac{\theta}{(2+s)\,(c^2-1)^{1+\frac{1}{2}s}\log c} -$$

$$-\frac{2c^{\frac{1}{2}-s}}{\log c}\left[\sum_\beta \frac{\sin(\beta\log c)}{\beta} + \left(\frac{1}{2}-s-\frac{1}{\log c}\right)\sum_\beta \frac{\cos(\beta\log c)}{\beta^2} + \varrho\sum_\beta\frac{1}{\beta^3}\right].$$

and in particular we have for $s=0$

$$G_0(c) = -\log 2 + Li(c) + \frac{\theta'}{2\,(c^2-1)\log c} -$$

$$-\frac{2c^{\frac{1}{2}}}{\log c}\left[\sum_\beta \frac{\sin(\beta\log c)}{\beta} + \left(\frac{1}{2}-\frac{1}{\log c}\right)\sum_\beta \frac{\cos(\beta\log c)}{\beta^2} + \varrho'\sum_\beta\frac{1}{\beta^3}\right]$$

In these equations we have got expressed as trigonometrical series the terms the occurrence of which makes it nearly impossible to arrive at a direct and complete determination of either $G_s(c)$ or $G_0(c)$. All we know about these series, is that

$$\sum_\beta \frac{\cos(\beta\log c)}{\beta^2} \text{ and } \sum_\beta \frac{1}{\beta^3}$$

converge unconditionally and that their values are rather small, because we have

$$\sum_\beta \frac{1}{\beta^2} = 0.023105, \quad \sum_\beta \frac{1}{\beta^3} < 0.002.$$

Further, that

$$\sum_\beta \frac{\sin(\beta\log c)}{\beta}$$

is a discontinous function of c suddenly changing its value, each time its argument becomes equal to the first, second, third,... power of a prime number.

This suggests that we eliminate the discontinuous function between the two equations and merely retain the relation

$$\left[G_s(c) - \log|\zeta(s)| - Li(c^{-s+1})\right] - c^{-s}\left[G_0(c) + \log 2 - Li(c)\right] =$$

$$= \frac{2\,s\,c^{\frac{1}{2}-s}}{\log c}\left[\sum_\beta \frac{\cos(\beta \log c)}{\beta^2} + A\sum_\beta \frac{1}{\beta^3}\right] + \frac{B}{c^{2+s}\log c}.$$

Whatever may be here the values of the coefficients A and B, from what precedes we may infer that they are finite and rather small, so that for $s > \frac{1}{2}$ and for tolerably large values of c the right-hand side tends rapidly to zero.

Regarding it as a vanishing quantity we are led to conclude that the relation

$$\left[G_s(c) - \log|\zeta(s)| - Li(c^{-s+1})\right] = c^{-s}\left[G_0(c) + \log 2 - Li(c)\right]$$

furnishes approximatively the value of $G_s(c)$ as soon as $G_0(c)$ be given, that is, as soon as we know how many prime numbers and powers of prime numbers are to be found among the integers not exceeding c.

The last equation necessarily takes a slightly altered form when s is tending to unity. In that case we must make use of the expansions

$$\log|\zeta(s)| = -\log(s-1) + (s-1)P_1(s-1),$$

$$Li(c^{-s+1}) = C + \log\log c^{s-1} + (s-1)P_2(s-1),$$

from which we have ultimately

$$\lim_{s=1}\left[\log|\zeta(s)| + Li(c^{-s+1})\right] = C + \log\log c.$$

Hence the value of $G_1(c)$ may be derived approximately from

$$\left[G_1(c) - C - \log\log c\right] = c^{-1}\left[G_0(c) + \log 2 - Li(c)\right].$$

Moreover it is evident that a relation similar to that between

$G_s(c)$ and $G_0(c)$ exists between two integrals $G_s(c)$ and $G_t(c)$, and that for $s > t$ and $s > \frac{1}{2}$ we may write

$$\left[G_s(c) - \log|\zeta(s)| - Li(c^{-s+1})\right] =$$

$$= c^{-s+t}\left[G_t(c) - \log|\zeta(t)| - Li(c^{-t+1})\right].$$

Lastly, we may remark that it is perfectly admissible to differentiate with respect to s the equation connecting $G_s(c)$ and $G_0(c)$.

Remembering that

$$-\left[\frac{d}{ds}G_s(c)\right]_{s=0}$$

is equal to the logarithm of the least common multiple $M(c)$ of all integers less than c, and that $\zeta'(0):\zeta(0) = \log 2\pi$, we find, by putting s equal to zero after the performance of the differentiation,

$$\left[\log M(c) + \log 2\pi - c\right] - \log c\left[G_0(c) + \log 2 - Li(c)\right] =$$

$$= -\frac{2c^{\frac{1}{2}}}{\log c}\left[\sum_\beta \frac{\cos(\beta \log c)}{\beta^2} + A\sum_\beta \frac{1}{\beta^3}\right] + \frac{B'}{c^3 \log c}.$$

Now although the second member increases as c increases, it remains relatively small with respect to c and $\log M(c)$; therefore we may expect the relation

$$\left[\log M(c) + \log 2\pi - c\right] = \log c\left[G_0(c) + \log 2 - Li(c)\right]$$

to furnish approximately the value of $\log M(c)$.

The following test-cases abundantly show, that already for a c of moderate magnitude the approximation is very close.

I. $c = e^2 = 7.389$, $G_2(c) = 0.45277$, $G_3(c) = 0.18077$.

$$[G_3(c) - \log\zeta(3) - Li(e^{-4})] = 0.00052,$$
$$e^{-2}[G_2(c) - \log\zeta(2) - Li(e^{-2})] = 000054.$$

II. $c = e^3 = 20.086$, $G_1(c) = 1.69330$, $G_2(c) = 0.48456$.

$$[G_2(c) - \log\zeta(2) - Li(e^{-3})] = 0.00089,$$
$$e^{-3}[G_1(c) - C - \log 3] = 0.00088.$$

III. $c = e^5 = 148.413, \quad G_0(c) = 38.50953, \quad G_1(c) = 2,18005,$
$\log M(c) = 141.66097.$

$$[G_1(c) - C - \log 5] = - 0.00661,$$
$$e^{-5}[G_0(c) + \log 2 - Li(e^5)] = - 0.00662.$$

$$[\log M(c) + \log 2\pi - e^5] = - 4.914,$$
$$\log e^5 [G_0(c) + \log 2 - Li(e^5)] = - 4.913.$$

IV. $c = e^7 = 1096.633, \quad G_0(c) = 191.79563, \quad G_1(c) = 2.52401.$

$$[G_1(c) - C - \log 7] = 0.00088,$$
$$e^{-7}[G_0(c) + \log 2 - Li(e^7)] = 0.00090.$$

If we have $s > 1$, it is fairly evident that for large values of c already the expression

$$[G_s(c) - \log \zeta(s) - Li(c^{-s+1})]$$

itself may be considered as evanescent, and that by equating it to zero we are sure to obtain, quite independent of the value of $G_0(c)$, a result for $G_s(c)$ that involves only a very small error.

But if we had $s = 1$, or even $\frac{1}{2} < s < 1$, this result would become unreliable, so that for $s \leqq 1$ a previous knowledge of the value of $G_0(c)$ remains indispensable.

This is connected with the fact that in the latter case $G_s(c)$ diverges as c becomes infinite.

Indeed we have for $s = 1$

$$G_1(c) = \sum_{p<c} p^{-1} + \sum_{n=2}^{n=\infty} \frac{1}{n} \sum_{p^n<c} p^{-n},$$

and it is the first term $\sum_{p<c} p^{-1}$ that increases beyond all limits for $c = \infty$ [1]).

We will examine more clocely the behaviour of $\sum_{p<c} p^{-1}$.

As s still surpasses $\frac{1}{2}$, we may write.

$$\lim_{c=\infty} [G_1(c) - C - \log \log c] = 0,$$

or

[1]) This was already stated by EULER ("Introductio in Analysin infinitorum," I, § 279).

$$\lim_{c=\infty}\left[\sum_{p<c} p^{-1} - \log\log c\right] = C - \sum_{n=2}^{n=\infty} \frac{1}{n} \sum p^{-n}.$$

The numerical constant at the right hand side is evaluated as follows. We suppose $s > 1$ and consider again the relation

$$\log \zeta(s) = \sum_1^{\infty} \frac{1}{n} \sum p^{-ns}.$$

From it we have conversely

$$\sum p^{-s} = \sum_{h=1}^{h=\infty} \frac{(-1)^{\mu_h}}{h} \log \zeta(hs) \ ^{1)},$$

where h denotes the successive terms of the infinite sequence

$$1,\ 2,\ 3,\ 5,\ 6,\ 7, \ldots\ldots,$$

formed by writing in ascending order the integers not admitting multiple factors, and μ_h stands for the number of prime factors of h (this number h itself, if prime, included).

Thus making s tend to unity we have simultaneously

$$\lim_{s=1}\left[\sum p^{-s} - \log \zeta(s)\right] = -\sum_{n=2}^{n=\infty} \frac{1}{n} p^{-n},$$

$$\lim_{s=1}\left[\sum p^{-s} - \log \zeta(s)\right] = \sum_{h=2}^{h=\infty} \frac{(-1)^{\mu_h}}{h} \log \zeta(h) = -0.31572,$$

and at once find

$$\sum_{n=2}^{n=\infty} \frac{1}{n} \sum p^{-n} = 0.31572,$$

so that finally

$$\lim_{c=\infty}\left[\sum_{p<c} p^{-1} - \log\log c\right] = C - 0.31572 = 0.26150.$$

We mentioned this result in order to compare it with a similar

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[1]) From this formula it is readily seen 1st that Σp^{-s} is an analytic function $f(s)$ of s, we may affirm to exist in the right half-plane, 2nd that the band between the parallels $s = \frac{1}{2} + in$ and $s = 0 + in$ is dotted with an infinity of logarithmic discontinuities of $f(s)$, 3rd that these discontinuities grow more and more dense as we approach the axis of imaginaries, so as to prevent $f(s)$ effectually from being continued into the left half-plane.

though wholly empirical formula communicated by LEGENDRE [1]). According to LEGENDRE we shall have

$$\sum_{p<c} p^{-1} = \log(\log c - 0.08366) - 0.22150\,,$$

but LEGENDRE not taking 2 for a prime number, as we did, 0,5 should be added to his result, so that we must read

$$\sum_{p<c} p^{-1} = \log(\log c - 0.08366) + 0.27850\,,$$

or ultimately for $c=\infty$

$$\lim_{c=\infty}\left[\sum_{p<c} p^{-1} - \log\log c\right] = 0.27850\,.$$

Thus it appears that the error in LEGENDRE's formula for $c=\infty$ amounts to 0.017, this error diminishing somewhat, when the formula is used for large though finite values of c.

In the preceding we have considered almost exclusively the integrals $G_s(c)$; we now proceed to show that we may deal similarly with the sum $\sum_{p<c} p^{-s}$, though by so doing the formulae lose in simplicity.

From

$$\sum p^{-(z+s)} = \sum_{h=1}^{h=\infty} \frac{(-1)^{\mu_h}}{h} \log\zeta(hz+hs)$$

we derive

$$\sum_{p<c} p^{-s} = \sum_{h=1}^{h=\infty} \frac{(-1)^{\mu_h}}{h} \cdot \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{c^z}{z}\log\zeta(hz+hs)\,dz = \sum_{h=1}^{h=\infty} \frac{(-1)^{\mu_h}}{h}\, G_{hs}\left(c^{\frac{1}{h}}\right).$$

But since

$$G_t(b) = 0\,,$$

if the limit b be less than 2, the foregoing infinite series actually contains only a finite number of terms and the summation letter h throughout satisfies the inequality

$$h < \frac{\log c}{\log 2}\,.$$

1) "Essai sur la théorie des nombres," 1808, p. 394. D'une loi remarquable observée dans l'énumération des nombres premiers.

Suppose that h successively becomes equal to

$$1,\ 2,\ 3,\ \ldots\ldots\ h',\ h'',\ \ldots\ldots\ h''',$$

where h' and h'' denote any pair of consecutive values of h, and consider the expression

$$H_s(c) = \sum_{p<c} p^{-s} - \sum_{h=1}^{h=h'} \frac{(-1)^{\mu h}}{h}\left[log\,|\,\zeta(s)\,| + Li\,(c^{-s+\frac{1}{h}})\right] -$$

$$- \sum_{h=h'}^{h=h''} \frac{(-1)^{\mu h}}{h} G_{hs}(c^{\frac{1}{h}}) \ .$$

Obviously we may write also

$$H_s(c) = \sum_{h=1}^{h=h'} \frac{(-1)^{\mu h}}{h}\left[G_{hs}\,(c^{\frac{1}{h}}) - log\,|\,\zeta(s)\,| - Li(c^{-s+\frac{1}{h}})\right],$$

and, from what has been proved before, we infer that $H_s(c)$ approximately obeys the relation

$$H_s(c) = c^{-s}\,H_0(c)\,.$$

It is by means of this equation that we are able to find $\sum_{p<c} p^{-s}$ as soon as $\sum_{p<c} p^0$ be given.

The choice of h' and h'' is quite arbitrary. If desired we may take $h' = h'''$; in general it will be advisable to determine h'' by the condition that $c^{\frac{1}{h''}}$ is just a little less than 5, in order to avoid the application of the approximation formula for $G_{hs}(c^{\frac{1}{h}})$ in cases where $c^{\frac{1}{h}}$ is too small a number.

It will be seen from the following examples that the formula may be relied on, if a not too close approximation is required.

I. $c = e^5 = 148.413, \quad \sum_{p<c} p^0 = 34, \quad \sum_{p<c} p^{-1} = 1.87980.$

$$h' = 3, \qquad h'' = 5, \qquad h''' = 7.$$

$$H_0(c) = 34 - [-\,log\,2 + Li\,(e^5)] + \frac{1}{2}[-\,log\,2 + Li\,(e^{\frac{5}{2}})] +$$

$$+ \frac{1}{3}[-\,log 2 + Li\,(e^{\frac{5}{3}})] + \frac{1}{5}[1] + \frac{1}{6}[1] + \frac{1}{7}[1] = -\,1.08524.$$

$$H_1(c) = \sum_{p<c} p^{-1} - \left[C + \log 5\right] + \frac{1}{2}\left[\log \zeta(2) + Li(e^{-\frac{5}{2}})\right] +$$

$$+\frac{1}{3}\left[\log\zeta(3)+Li(e^{-\frac{10}{3}})\right]+\frac{1}{5}\left[\frac{1}{2^5}\right]-\frac{1}{6}\left[\frac{1}{2^6}\right]+\frac{1}{7}\left[\frac{1}{2^7}\right]=\sum_{p<c} p^{-1}-1.88701.$$

The relation

$$H_1(c) = e^{-5} H_0(c)$$

gives

$$\sum_{p<c} p^{-1} = 1.87970, \qquad \Delta = 0.00010.$$

II. $c = e^7 = 1096.633, \quad \sum_{p<c} p^0 = 183, \quad \sum_{p<c} p^{-1} = 2.21239.$

$$h' = 3, \quad h'' = 5, \quad h''' = 10.$$

$$H_0(c) = 183 - \left[-\log 2 + Li(e^7)\right] + \frac{1}{2}\left[-\log 2 + Li(e^{\frac{7}{2}})\right] +$$

$$+\frac{1}{3}\left[-\log 2+Li(e^{\frac{7}{3}})\right]+\frac{1}{5}\left[2+\frac{1}{2}\right]-\frac{1}{6}\left[2\right]+\frac{1}{7}\left[1\right]-\frac{1}{10}\left[1\right]=0.88308.$$

$$H_1(c) = \sum_{p<c} p^{-1} - \left[C + \log 7\right] + \frac{1}{2}\left[\log \zeta(2) + Li\left(e^{-\frac{7}{2}}\right)\right] +$$

$$+\frac{1}{3}\left[\log \zeta(3) + Li\left(e^{-\frac{14}{3}}\right)\right] + \frac{1}{5}\left[\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{2}\cdot\frac{1}{2^{10}}\right]-\frac{1}{6}\left[\frac{1}{2^6}+\frac{1}{3^6}\right]+$$

$$+\frac{1}{7}\left[\frac{1}{2^7}\right]-\frac{1}{10}\left[\frac{1}{2^{10}}\right]=\sum_{p<c} p^{-1} - 2.21162.$$

The relation

$$H_1(c) = e^{-7} H_0(c)$$

gives

$$\sum_{p<c} p^{-1} = 2.21243, \quad \Delta = 0.00004.$$

III. $c = e^4 = 54.598, \quad \sum_{p<c} p^0 = 16, \quad \sum_{p<c} p^{-3} = 0.17472.$

$$h' = 2, \quad h'' = 3, \quad h''' = 5.$$

$$H_0(c) = 16 - \left[-\log 2 + Li(e^4)\right] + \frac{1}{2}\left[-\log 2 + Li(e^2)\right] +$$

$$+\frac{1}{3}\left[2\right]+\frac{1}{5}\left[1\right]=0.05949.$$

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$$H_3(c) = \sum_{p<c} p^{-3} - \left[\log \zeta(3) + Li(e^{-8})\right] + \frac{1}{2}\left[\log \zeta(6) + Li(e^{-10})\right] +$$

$$+ \frac{1}{3}\left[\frac{1}{2^9} + \frac{1}{3^9}\right] + \frac{1}{5}\left[\frac{1}{2^{15}}\right] = \sum_{p<c} p^{-3} - 0.17472.$$

The relation

$$H_3(c) = e^{-12} H_0(c)$$

gives

$$\sum_{p<c} p^{-3} = 0.17472, \quad \Delta = 0.00000.$$

Chemistry. — "*Thermodynamics of Standard Cells*" (First Part). By Dr. ERNST COHEN (Communicated by Prof. H. W. BAKHUIS ROOZEBOOM).

1. As the elegant researches on the CLARK and WESTON cells performed by KAHLE, JAEGER and WACHSMUTH [1]) in the Physikalisch-Technische Reichsanstalt have now been brought to a close and their measurements have been verified by the highly accurate determinations of CALLENDAR and BARNES [2]), it appeared to me worth while to check these experimental results by thermodynamics.

Not only is the E. M. F. of the CLARK-cell used as the unit of E. M. F., but the value of the exact knowledge of that standard is much increased since by the measurements of CALLENDAR and BARNES [3]), the value of the heat-unit is based on it.

During the research which I shall communicate in the following lines, it has appeared to me that the accepted views regarding the mechanism of the action of the normal cells are incorrect because the old mistake has again been made of ignoring the actual phases of the substances which take part in the changes within the cell. This neglect has already given rise to erroneous calculations and conclusions.

These erroneous views will first be discussed and then a complete theory of the action of the normal cells will be given. We will

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[1]) KAHLE, Zeitschrift für Instrumentenkunde, **12**, S. 117 (1892); ibid. **13**, S. 191 S. 293 (1893). WIEDEMANN's Annalen **51**, S. 174 und 203 (1894), ibid **64**. S. 92 (1898); JAEGER und WACHSMUTH, Elektrotechn. Zeitschrift **15**. S. 507 (1894); WIEDEMANN's Annalen **59**. S. 575 (1896); W. JAEGER. Elektrotechn. Zeitschrift **18**. S. 647 (1897): WIEDEMANN's Ann. **63**. S. 354 (1897); JAEGER und KAHLE, Zeitschrift für Instrumentenkunde (1898) 161.

[2]) Proc. Roy. Society **62**. 117.

[3]) Rep. Brit. Association 1899. Section A. Physical Review, Vol. X No. 4, p. 202. April 1900.