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The following papers were read:

Mathematics. — „*On the cyclographic space representation of Joachimsthal's circles.*" By Prof. P. H. SCHOUTE.

1. In his „Cyklographie" Dr. W. FIEDLER has developed a theory, in which any circle of the plane is represented in space by one of two points of the normal, erected in its centre on the plane, and having on either side a distance from this centre equal to the radius. The ambiguity of this representation can be useful in the distinction of the two senses, in which a point can move along the circle. This is not necessary here.

According to the FIEDLERIAN representation the right cone, whose vertex is a point P of the plane of the circles, whose axis is the normal in P on this plane, and whose vertex angle is a right one, corresponds to the net of the circles passing through P. Likewise the pencil of the circles passing through P and Q has as image a rectangular hyperbola situated in the plane bisecting $P\,Q$ orthogonally; of this hyperbola the line common to its plane and the plane of the circles is the imaginary axis, the intersection of this line with PQ is the centre, and the normal in this point to the plane of the circles is the real axis. Or, put more generally: the circles of a pencil are represented by a rectangular hyperbola, whose real or imaginary axis is normal to the plane of the circles according to the points common to the circles being real or imaginary, and which breaks up into two straight lines, if these points coincide And the circles of a general net are represented by a rectangular hyperboloid with one or two sheets, according to the circle cutting the circles of the net orthogonally being real or imaginary.

In different cases the FIEDLERIAN theory can give a clear and concise idea of the position of ranges of circles. So a. o. the circles having double contact with a given ellipse ε. If this ellipse ε be represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$, the ellipse $\frac{x^2}{a^2 - b^2} + \frac{z^2}{b^2} = 1, y = 0$ and the hyperbola $-\frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2} = 1,\ x = 0$ correspond to the two ranges of circles having double contact with ε. As is easily explained these curves are transformed into the focal conics of ε by inverting the sign of z^2.

In the following lines we study the surface that forms the image of the twofold infinite system of JOACHIMSTHAL's circles of ε.

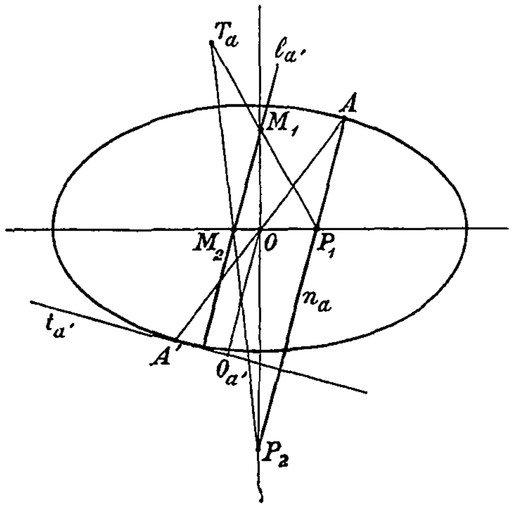
2. Through any point P of the plane of ε can be drawn four normals to ε. The footpoints A, B, C, D of these normals may be called „conormal". If the point diametrically opposite to A on ε is indicated by A', the known theorem of JOACHIMSTHAL says that A', B, C, D are concyclic, if A, B, C, D are conormal.

This non-reversible theorem has been completed by LAGUERRE in remarking that the circle $A'BCD$ meets the tangent $t_{a'}$ in A' to ε for the second time in the projection $O_{a'}$ of the centre O of ε on $t_{a'}$. In other words:

If P describes the normal n_a in A to ε, the corresponding circles $A'BCD$ form a pencil, as all these circles pass through A' and $O_{a'}$. This pencil being represented by a rectangular hyperbola, the image in question is the locus of a simply infinite number of rectangular hyperbolae. However, before we proceed to the deduction of this surface, we investigate somewhat more closely the correspondence between the points P of the normal n_a and the centres M on the line $l_{a'}$ bisecting orthogonally the segment $A'O'_a$.

3. The relation between the points P and M on n_a and $l_{a'}$ is a (1,1) correspondence, i. e. these points describe projective ranges. If P

is at infinity on n_a, this is also the case with M on $l_{a'}$. So the point at infinity common to n_a and $l_{a'}$ corresponds to itself, i. e. the projective ranges are in perspective. The centre of perspective T_a is immediately found as the point common to the joints $P_1 M_1$ and $P_2 M_2$ (fig 1) of the pairs of corresponding points (P_1, M_1) and (P_2, M_2). This point being found, it is possible to indicate the centre M of the circle of JOACHIMSTHAL corresponding to any point P of n_a.



Analytically the obtained results are given back as follows. The coordinates of A being $a \cos \varphi$, $b \sin \varphi$, the equations of the right lines n_a, $l_{a'}$, $P_1 M_1$, $P_2 M_2$ are successively:

$$n_a \dots \frac{ax}{\cos \varphi} - \frac{by}{\sin \varphi} = c^2, \quad l_{a'} \dots \frac{-2\,ax}{\cos \varphi} + \frac{2\,by}{\sin \varphi} = c^2 ,$$

$$P_1 M_1 \dots \frac{ax}{\cos \varphi} + \frac{2\,by}{\sin \varphi} = c^2, \quad P_2 M_2 \dots \frac{-2\,ax}{\cos \varphi} - \frac{by}{\sin \varphi} = c^2 .$$

So the centre of perspective T_a has the coordinates $-\frac{c^2}{a} \cos \varphi$, $\frac{c^2}{b} \sin \varphi$ and this point describes the ellipse ε', the four vertices of which are the four real cusps of the evolute of ε. And the joints $A T_a$ and $A' T_a$ likewise envelope ellipses, etc.

4. The simple relation between the lines n_a and $l_{a'}$ proves immediately that $l_{a'}$ is normal to the ellipse $\frac{4\,x^2}{a^2} + \frac{4\,y^2}{b^2} = 1$ and that through any point M pass four of these normals $l_{a'}$. In other words, any normal to the plane of the circles meets four of the rectangular hyperbolae and so contains eight points of the locus. As the point at infinity common to all these normals does not lie on one of the rectangular hyperbolae, the locus is a surface of the eighth order. We confirm this result by the deduction of its equation.

The cones that form the images of all the circles through A' and all the circles through $O_{a'}$ are represented by

1*

$$\left.\begin{array}{l}(x - a\cos\varphi)^2 + (y - b\sin\varphi)^2 = z^2 \\ \left(x - \dfrac{ab^2\cos\varphi}{b^2\cos^2\varphi + a^2\sin^2\varphi}\right)^2 + \left(y - \dfrac{a^2b\sin\varphi}{b^2\cos^2\varphi + a^2\sin^2\varphi}\right)^2 = z^2\end{array}\right\} .$$

So these equations represent together the rectangular hyperbola projecting itself in $l_{a'}$. Putting for simplicity's sake u^2 for $x^2 + y^2 - z^2$, these equations can be reduced to

$$\left.\begin{array}{l}u^2 + b^2 + c^2\cos^2\varphi - 2\,ax\cos\varphi = 2\,by\sin\varphi \\ u^2(a^2 - c^2\cos^2\varphi) - 2\,ab^2x\cos\varphi + a^2b^2 = 2\,a^2by\sin\varphi\end{array}\right\} .$$

So elimination of $\sin\varphi$ gives for $\cos\varphi$ the relation

$$\cos\varphi\,[(u^2 + a^2)\cos\varphi - 2\,ax] = 0 ,$$

which breaks up into

$$\cos\varphi = 0 , \qquad \cos\varphi = \frac{2\,ax}{u^2 + a^2} .$$

As φ is variable, the first condition cannot serve here. It corresponds in fact to this, that the two cones with the vertices A' and $O_{a'}$ coincide instead of determining together a rectangular hyperbola, when A' is one of the extremities of the minor axis of ε; whilst a similar treatment, in which the parts of $\cos\varphi$ and $\sin\varphi$ are inverted, leads to the relation $\sin\varphi = 0$, corresponding in the same manner to the coincidence of these cones, if A' is one of the extremities of the major axis of ε. Substitution of the other value of $\cos\varphi$ in the first of the second pair of cone equations gives the result in the form

$$\frac{4\,a^2x^2}{(u^2 + a^2)^2} + \frac{4\,b^2y^2}{(u^2 + b^2)^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

which really represents a surface of the eighth order.

Inversely this simple equation shows that the surface represented by it may be generated by rectangular hyperbolae, by considering it as the result of the elimination of ψ between

$$\left.\begin{array}{l}2\,ax = (u^2 + a^2)\cos\psi \\ 2\,by = (u^2 + b^2)\sin\psi\end{array}\right\} ,$$

which represent for any constant value of ψ a rectangular hyperbola lying in the plane

$$\frac{2\,ax}{\cos\psi} - \frac{2\,by}{\sin\psi} = c^2 .$$

For $\psi = \varphi \pm 180°$ this equation transforms itself into that of $l_{a'}$.

5. We signalize here some particularities of the found surface.

a. The intersection of the cone $x^2 + y^2 - z^2 = 0$ with the plane at infinity is a fourfold curve of the surface. For the substition of

$$x = p + z \cos\lambda\,, \qquad y = q + z \sin\lambda$$

into the equation 1) yields an equation of the fourth degree for z. It is reduced to a cubic equation under the condition

$$(p \cos\lambda + q \sin\lambda)^2 = a^2 \cos^2\lambda + b^2 \sin^2\lambda$$

and to a quadratic equation for

$$p \cos\lambda + q \sin\lambda = 0.$$

So the four tangent planes in the point $x = z \cos\lambda$, $y = z \sin\lambda$ at infinity are represented by

$$x \cos\lambda + y \sin\lambda - z = \pm\sqrt{a^2 \cos^2\lambda + b^2 \sin^2\lambda}\,,$$
$$x \cos\lambda + y \sin\lambda - z = 0\,,$$

the last of these counting twice. The deduction of the envelopes of these planes for various values of λ shows that the surface 1) is touched at infinity by the developable surface or torse circumscribed to the tangential pencil of quadrics, to the four quadrics flattened to conics of which belong the fourfold conic of 1) and the ellipse ε; moreover it is osculated at infinity by the cone $x^2 + y^2 - z^2 = 0$. So this cone intersects the surface 1) in a curve in space of the sixteenth order to which the fourfold conic of 1) belongs six times. The completing curve in space lies on the cylinder $\frac{4x^2}{a^2} + \frac{4y^2}{b^2} = 1$, in which 1) is transformed for $u^2 = 0$; this cylinder meets the surface 1) in another curve of the twelfth order, etc.

b. The intersection of the surface 1) with each of the planes ZOX, ZOY consists of four straight lines and a rectangular hyperbola counting twice. So $y = 0$ yields the four lines $x \pm z = \pm a$ and the hyperbola $x^2 - z^2 + a^2 = 0$, and likewise $x = 0$ yields the four lines $y \pm z = \pm b$ and the hyperbola $y^2 - z^2 + b^2 = 0$. So 1) contains besides the fourfold conic at infinity still two double conics; moreover it bears eight right lines, viz. the four pairs of lines into which the rectangular hyperbolae of the vertices of ε are degenerated.

c. The surface contains four separate double points, the four vertices of ε. By transporting the axes of coordinates parallel to themselves to one of these points as origin and equalling the terms of the second order to zero the equation of the corresponding osculating cone is found.

d. The curve limiting the projection of 1) on the plane of the circles is obtained by elimination of z between 1) and its differential quotient according to z. It consists of the intersection of 1) with the plane of the circles and of the projection of a curve in space. The first is the locus of the octuples of points common to the corresponding pairs of circles of the two circle involutions

$$\left.\begin{array}{r}(x \pm a \sec \lambda)^2 + y^2 = a^2 \, tg^2 \, \lambda \\ x^2 + (y \pm b \operatorname{cosec} \lambda)^2 = b^2 \cot^2 \lambda\end{array}\right\}$$

and as such a quadricircular octavic; the isolated double points in the vertices of ε excepted, all its points are imaginary. The second is found by the elimination of u^2 between 1) and its differential quotient according to u^2, which, v being substituted for u^2, comes to the elimination of v between

$$\frac{4\, a^2\, x^2}{(v + a^2)^2} + \frac{4\, b^2\, y^2}{(v + b^2)^2} = 1,$$

$$\frac{4\, a^2\, x^2}{v + a^2} + \frac{4\, b^2\, y^2}{v + b^2} = 2\, v + a^2 + b^2.$$

By solution we find

$$(v + a^2)^3 = 4\, a^2\, c^2\, x^2 , \quad (v + b^2)^3 = -\, 4\, b^2\, c^2\, y^2$$

and by elimination of v

$$(2\, a\, x)^{\frac{2}{3}} + (2\, b\, y)^{\frac{2}{3}} = c^{\frac{4}{3}} \;,$$

i. e. the evolute of the ellipse $\frac{4\,x^2}{a^2} + \frac{4\,y^2}{b^2} = 1$. This result was to be foreseen. For the normal at the plane of the circles in a point of this evolute meets two immediately succeeding rectangular hyperbolae and is therefore tangent to the surface on either side of the plane $X\,O\,Y$. The curve of contact itself, of which this evolute is the projection, is of the twelfth order. The cylinder of the sixth order, of which this evolute is a right section, meets the surface 1) moreover in a curve in space of the order 24.

6. The found image 1) can be useful in different researches about the system S of the circles of JOACHIMSTHAL. By determining the number of points common to this surface and a rectangular hyperbola, a parabola and a straight line, we find that the system S has the characterizing numbers 4,8,16; in other words, it contains four circles passing through two given points, eight circles passing through a given point and touching a given line, sixteen circles touching two given lines. In the same manner is proved that it contains sixteen circles touching two given circles, etc.

7. If we are given a parabola instead of an ellipse, all the circles passing through three conormal points pass also through the vertex of the parabola. Here the found surface of the eighth order is reduced to the right cone $x^2 + y^2 = z^2$, of which the vertex of the parabola $y^2 = 2px$ is the vertex. And the case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $z = 0$ leads to the surface

$$\frac{4a^2x^2}{(u^2+a^2)^2} - \frac{4b^2y^2}{(u^2-b^2)^2} = 1$$

and is quite analogous to that of the ellipse.

Physics. — *„On maxima and minima of apparent brightness resulting from optical illusion."* By Dr. C. H. WIND. (Communicated by Prof. H. HAGA).

1. If we see on a surface two zones of different (real) brightness united by a transition-zone whose brightness decreases continuously from the brighter down to the darker zone, this transitionzone *seems* to be separated from the brighter zone by a still brighter line (maximum of brightness) and from the darker zone by a still darker line (minimum of brightness).

2. This phenomenon, which — as will be seen from what follows — presents itself under very different kinds of conditions was first observed by me in a drawing carefully and succesfully executed by Mr. VAN GRIEKEN, of the firm VAN DE WEYER at Groningen, by means of lithography. This drawing of which fig. 1 is a photographic reproduction (reduced to $^1/_4$ of its size), which unsatisfactory as it is, yet enables us to observe the phenomenon, consists of a great number of parallel lines of equal thickness drawn at intervals of 1 m.M. in two outer zones, at intervals of 0.4 m.M. in a middlezone,