Mathematics. — "*The representation of the Screws of* BALL *passing through a point or lying in a plane, according to the method of* CAPORALI." By Prof. J. CARDINAAL.

1. This communication must be regarded as a continuation and enlargement of a lecture delivered at the 70th Congress of the "Gesellschaft deutscher Naturforscher und Aerzte" at Dusseldorf (Sept. 1898) and published in the last "Jahresbericht der Deutschen Mathematiker-Vereinigung". There the method applied in the following pages has been considered in its relation to the "Theory of Screws" by Sir R. St. BALL, so I think I can suffice by beginning with a few brief indications indispensable for the understanding of the purpose of the communication.

2*a*. The motion of a body considered here is the motion with freedom of the 4th degree; the screws about which motion is possible form a quadratic complex, consisting of all screws reciprocal to a given cylindroid C^3.

b. We construct the screws passing through a point P and belonging to the complex by drawing perpendiculars through P to the generatrices of C^3; each of these perpendiculars moreover intersects two generatrices of C^3, equidistant from the middle plane (conjugate lines). The locus of these screws is the cone P^2.

c. In a similar manner we construct the screws situated in a plane π. They envelop a parabola π^2.

3. The representation of the rays of a quadratic complex has been treated among others by R. STURM and CAPORALI. We find it inserted at large in Mr. STURM's "Liniengeometrie", III, pages 272 - 282. The special complex formed by the screws alluded to belongs to the type treated on pages 438—444. Although the results laid down in the following correspond with those obtained there, as could be expected, there is a great difference in the investigation; this difference can be circumscribed as follows:

1st. The proofs are here deduced immediately from the theory of BALL, whereas with STURM they follow as special cases out of the complex.

2nd. The constructions, more particularly a principal construction, are really executed.

4. Fig. 1 represents the axonometrical projection of a cylindroid, whose construction is understood to be known. The nodal line d coincides with the axis OZ; we suppose further that the rotation

of the generatrices has begun in the plane $Z O Y$; on the line O—0 the maximum pitch g has been measured out; on the following lines I—1, II—2, etc. the succeeding pitches are now continually measured out from the nodal line, the measuring ceasing in the position where the rotation amounts to 180°, thus the whole height of the cylindroid having been described on both sides of the centre O. To explain this we make use in the first place of the circle, which has served to find the length of the pitch of any generatix, drawn on half the scale of the principal figure, where the length of any generatrix (e. g. II—2) is indicated with its corresponding angle of deviation.

In the second place we see axonometrically constructed the pitch curve projected on $X O Y$ with the projections O—0, O—1 etc. of the generatrices. Further is drawn the perpendicular $A B$ to II—4 passing through A on $a \equiv 1$—7. According to 2*b*. this is a screw, so it meets the line $a' \equiv I$—5 conjugate to 1—7.

Remarks. *a*. The projection of the above mentioned pitch curve lies entirely on one side of the axis $O Y$. Evidently this is half the figure we obtain in constructing the curve with the equation: $\varrho = a + 2 r \cos^2 \theta$. It is merely a consequence of our peculiar manner of measuring that by the followed construction only half the figure is obtained.

b. In the constructed figure all the pitch values have the same sign: if this were not the case, the figure of the projection would be changed, the curve half drawn, half dotted would show a double point, so the entire projection a fourfold point. This last has now become isolated.

5. Fig. 2 represents the parallel projection of the principal curve of the representation of Caporali and must be considered in connection with fig. 1. On the generatrix a of C^3 a point A has been fixed. Through A the screw $A A' B$ has been drawn, being one of the rays of the pencil through the centre A in the plane $A a' \equiv \alpha$. A, as pole of α, determines with α a linear system of the 3d order of linear complexes.

With these figures we suppose that C^3 lies in the space Σ, the principal curve in the conjugate space Σ_1. To any linear complex in Σ a plane in Σ_1 corresponds, to any screw a point. The principal curve is the locus of the points in Σ_1 to which corresponds in Σ not only a single screw but a pencil of screws. According to these conventions and to the indication sub N°. 1

we can pass to the analysis of the principal curve. It consists of the following parts:

a. The conic K^2_d in plane δ_1, locus of the points corresponding to pencils of screws having a ray in common with the pencil (A, α) and whose planes are parallel to the nodal line d.

b. The conic K^2_u in plane υ_1, locus of the points corresponding to pencils of screws having in common a ray with (A, α) and consisting of parallel rays.

c. The line $l_1 \equiv D_d D_u$, intersecting K^2_d and K^2_u, locus of the points corresponding to pencils of screws whose vertices lie on a' and whose planes pass through a.

Both conics have two imaginary points in common; the planes δ_1 and υ_1 are the loci of the points corresponding to screws parallel to d and to those in the plane at infinity. They have not been indicated here.

In the figure have further been constructed the vertices T_g and T_k of the two quadratic cones, determined by K^2_d and K^2_u. The line connecting T_g, T_k meets δ_1 and υ_1 in M_d and M_u.

6. Now the forms of Σ_1 corresponding to the cones and parabolas of Σ can be found. It is evident that we shall get curves in Σ_1. Let us take a vertex P and construct a cone of screws P^2. The construction gives rise to the following remarks:

a. Let us imagine through (A, α) a zero system, formed by two reciprocal systems of points and planes with the property that any point lies in its corresponding polar plane ("Nullsystem" of Moebius), with a linear complex σ situated in it; the polar plane of P intersecting cone P^2 in two generatrices, the corresponding plane σ_1 intersects in two points the curve P_1^2 corresponding to P^2; so P_1^2 is a conic.

b. One screw of P^2 intersects a and a'; so P_1^2 meets l_1 in one point.

c. One screw of P^2 is parallel to d; so P_1^2 intersects δ_1 in one point not situated on K^2_d.

d. It is very important to determine how many screws of P^2 belong to the pencils of parallel screws having one screw in common with (A, α). Let us bring a plane parallel to α through P; two screws of P^2, m and n, are parallel to the screws m' and n' of pencil (A, α); from this we conclude that m and m' are perpendicular to the same generatrix of C^3; so they belong to the same pencil of parallel screws, viz. to a pencil having with (A, α) one screw in common. The same can be said of n and n'; so P_1^2 intersects the conic K_u^2 in two points.

e. Further we must determine how many screws of P^2 belong to pencils of screws, whose planes are parallel to *d*, having with (A, α) a screw in common. Therefore it is necessary to determine the points common to screws of P^2 and the screws of (A, α) lying at the same time on a generatrix of C^3. Let us first consider the section of plane α with C^3; it consists of the line a' and a conic A^2. In the second place we must notice that the feet of the perpendiculars through P on the generatrices of C^3 form a conic B^2, lying in a plane having moreover with C^3 a right line b in common. Both degenerated cubic curves $A^2 + a'$, $B^2 + b$, intersect in three points. As the lines a' and b cannot intersect, those points lie in such a manner that a' meets once B^2, b once A^2 and A^2 once B^2. Now the last point L is the only point of intersection satisfying the above condition; so P_1^2 intersects K^2_d in *one* point.

7. The curves corresponding in Σ_1 with the screws, enveloping the parabolas in the planes π, are determined in the same way. We shall show this briefly.

a. Out of the pole of π with regard to the zero system through (A, α) two tangents can be drawn to the parabola π^2. So the corresponding curve π_1^2 is a conic.

b. One screw in π_1^2 meets a and a'; so π_1^2 intersects l_1 in one point.

c. One screw in π lies at infinity; so π_1^2 meets the plane v_1 in one point, not situated on K^2_u.

d. One screw of pencil (A, α) is parallel to the plane π; both belong to the same pencil of parallel screws; so π_1^2 has one point in common with K^2_u.

e. The line $\overline{\alpha\pi}$ common to α and π meets C^3 in its point of intersection with a' and moreover in two points M and N. Through M pass two tangents of π^2. One of these tangents contains a point lying on the generatrix m' of C^3, conjugate to that on which M is situated; the other one is perpendicular to m itself. The latter tangent determines with the screw AM a plane perpendicular to m; consequently in that plane there is a pencil having a screw in common with (A, α); this pencil belongs to those, whose plane is parallel to d. The same reasoning can be applied to N. Consequently the corresponding conic π_1^2 intersects the conic K^2_d in two points.

8. So the conics P_1^2 and π_1^2, having four points in common with the curve $K^2_u + K^2_d + l_1$, their number in Σ_1 would amount to ∞^4; there being however in Σ only ∞^3 cones and curves of the

complex, there must be a closer definition of the conics. This is also evident from the following. All cones P^2 of the complex belong to zero systems of a special kind. The line connecting P and A defines such a special linear complex of rays; it consists of all rays intersecting PA. All these systems have the point A in common; the net of rays through A contains the rays common to these systems and finally the screws common to all consist of the pencil (A, α) and the pencil through A perpendicular to a. Hence it follows:

All zero systems in which the screws of the cones of the complex are situated have in common the pencil through A perpendicular to a, to which in Σ_1 a point C_d on $K^2{}_d$ corresponds (compare STURM, III p. 276).

In a similar manner is proved that the pencil of screws common to all zero systems of the curves of the complex is the pencil of parallel screws lying in α, to which pencil a point C_u in Σ_1 on $K^2{}_u$ corresponds.

9. Now a construction will be deduced to find the points C_d and C_u. It is evidently sufficient if we consider the point of intersection of a plane, in which one of the conics $P_1{}^2$ or $\pi_1{}^2$ is lying, with $K^2{}_d$ and $K^2{}_u$; specially that point of intersection which is not at the same time a point of $P_1{}^2$ or $\pi_1{}^2$, for which we can choose in Σ a special cone or conic.

Let us take P on the nodal line d in the point where the generatrices with maximum and minimum pitch g and k meet. Cone P^2 breaks up into two planes through d; one plane is perpendicular to g, in which a pencil of screws lies with P as vertex, all screws having a pitch equal to that of k; the second is perpendicular to k, which also contains a pencil of screws having a pitch equal to that of g. To each of these pencils a right line corresponds in Σ_1; the first belongs to the cone $T^2{}_k$, the second to the cone $T^2{}_g$; further a screw of the degenerated cone P^2 coinciding with d, the corresponding plane of Σ_1 passes through $D_d\ T_g\ T_k$; so it intersects $K^2{}_d$ still in a point, the point C_d that was to be found.

10. From the preceding we easily deduce the construction of the point C_u. Let us bring plane π through g and k. The parabola π^2 breaks up into two pencils of parallel screws consisting of rays perpendicular to g and of rays perpendicular to k, the first having the pitch k, the second the pitch g. So the corresponding plane in Σ_1 passes through two generatrices of the cones $T^2{}_k$ and

CARDINAAL, The representation of the screws of BALL passing through a point or lying in a plane, according to the method of CAPORALI.

Cylindroid.

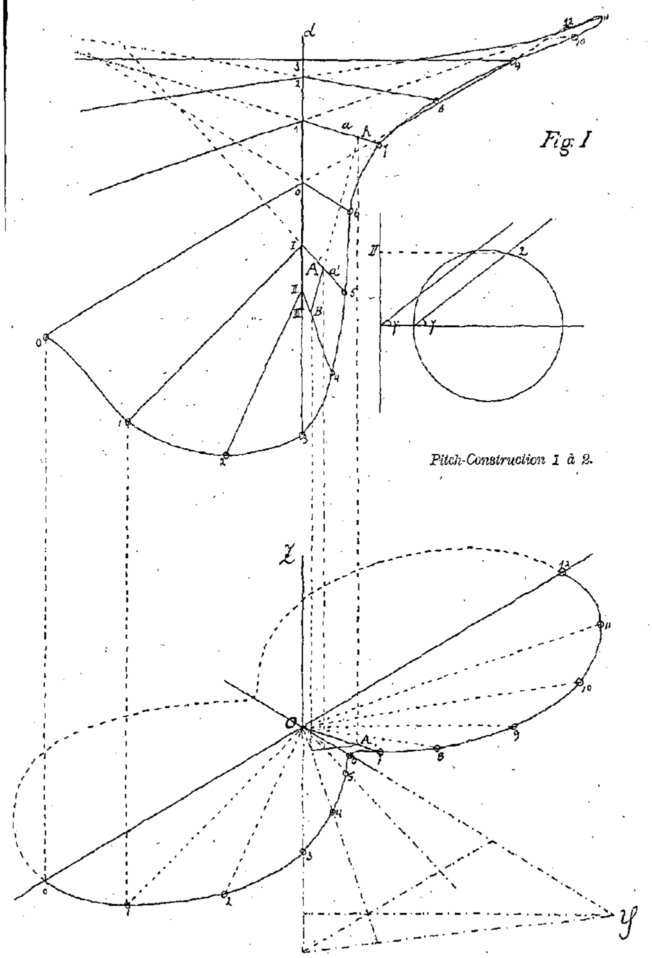

Fig. I

Pitch-Construction 1 à 2.

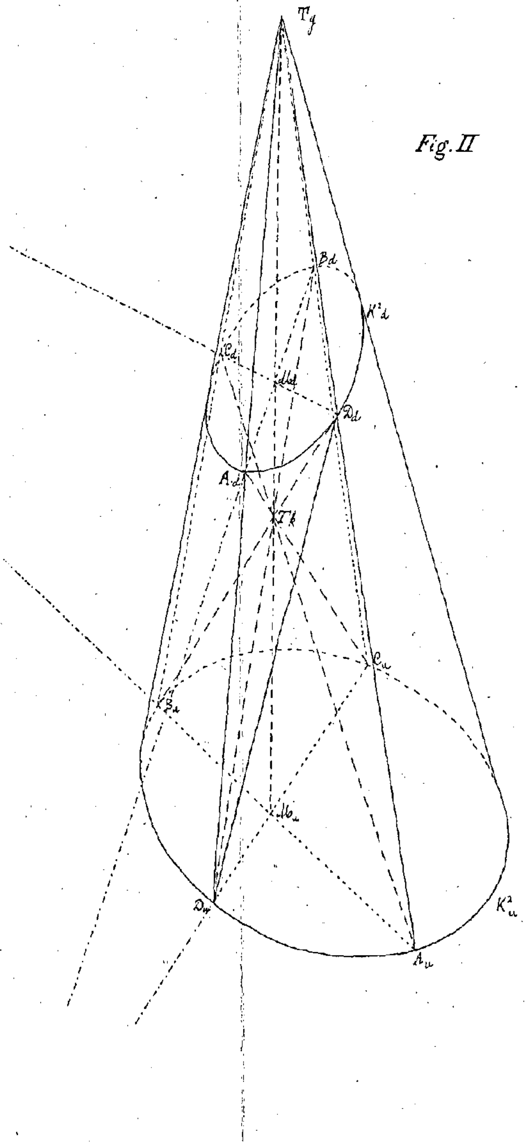

Fig. II

T^2_g and both pencils contain at the same time the ray at infinity of the plane $\overline{gk}$. The latter is a line through which the bitangential planes of C^3 pass; the point in Σ_1 corresponding to it is D_u. So the point C_u is found by constructing the plane $T^2{}_g T^2{}_k D_u$ and determining its point of intersection with K^5_u.

Physics. — *„On the vibrations of electrified systems, placed in a magnetic field. A contribution to the theory of the* ZEEMAN-*effect".* By Prof. H. A. LORENTZ.

(Will be published in the Proceedings of the next meeting.)

Mathematics. — *„On Trinodal Quartics".* By Prof. JAN DE VRIES.

1. If we consider the nodes D_1, D_2, D_3 of a trinodal plane quartic as the vertices of a triangle of reference, that curve has an equation of the form:

$$\Gamma_4 \equiv a_{11}\,x_2^2\,x_3^2 + a_{22}\,x_3^2\,x_1^2 + a_{33}\,x_1^2\,x_2^2 + $$

$$+ 2\,x_1\,x_2\,x_3\,(a_{12}\,x_3 + a_{23}\,x_1 + a_{31}\,x_2) = 0 \quad . \quad . \quad (1)$$

The equations

$$\Phi_2 \equiv b_1\,x_2\,x_3 + b_2\,x_3\,x_1 + b_3\,x_1\,x_2 = 0 \quad , \quad . \quad . \quad . \quad . \quad (2)$$

$$\Psi_2 \equiv c_1\,x_2\,x_3 + c_2\,x_3\,x_1 + c_3\,x_1\,x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

then represent two conics passing through the nodes.

If the coefficients of these equations satisfy the conditions

$$b_1\,c_1 = a_{11}\,, \quad b_2\,c_2 = a_{22}\,, \quad b_3\,c_3 = a_{33} \quad , \quad . \quad . \quad . \quad (4)$$

it is evident from the identity

$$\Phi_2\,\Psi_2 - \Gamma_4 \equiv x_1\,x_2\,x_3\,\Sigma\,(b_1\,c_2 + b_2\,c_1 - 2\,a_{12})\,x_3 \,. \quad . \quad . \quad (5)$$

that the two new couples of points common to Γ_4 and each of the two *associated* conics Φ_2 and Ψ_2 are situated on the right line corresponding to the equation

$$\Sigma\,(b_1\,c_2 + b_2\,c_1 - 2\,a_{12})\,x_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

18*