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Physics. — „*Calculation of the second correction to the quantity b of the equation of condition of* VAN DER WAALS." By Mr. J. J. VAN LAAR. (Communicated by Prof. J. D. VAN DER WAALS.)

In a paper, published in the Proceedings of the Meeting of the Section for Mathematics and Physics of the Royal Academy of Sciences 29th of Oct. 1898 (appeared Nov. 9th 1898), Prof. VAN DER WAALS has pointed out among others how a second correction to the quantity b of his equation of condition might be found. The integrations necessary to it, proving to be extremely tedious and lengthy, have not been calculated at full length.

I then tried to work out these integrations; I shall communicate in short the results found, referring, with respect to the various mathematical developments which led to my results, to a more extensive treatment that will shortly be published elsewhere (in the "Archives du Musée Teyler").

The form to be integrated [1]) ran as follows (see pages 142—143 of the cited Proceedings):

$$\iint \frac{N}{V} \cdot 2\pi (h + a \cos \theta)^2 \, dh \, d\theta \times \text{part of segment},$$

in which that part of segment is found to be:

$$\tfrac{1}{3} a^2 \sin \varphi \cos \varphi \sqrt{R^2 - a^2} + \tfrac{2}{3} R^3 \tan^{-1} \left(\tan \varphi \frac{\sqrt{R^2 - a^2}}{R} \right) -$$

$$- a \sin \varphi \left(R^2 - \tfrac{1}{3} a^2 \sin^2 \varphi\right) \tan^{-1} \frac{\sqrt{R^2 - a^2}}{a \cos \varphi}.$$

If we now first perform the integration with respect to θ between the limits 0 and θ_1, where θ_1 is given by the circumstance that the centre C of the third sphere cannot lie within the two spheres A and B (see fig. on page 142), we have to integrate:

$$2 \int_0^{\theta_1} (h + a \cos \theta)^2 \, d\theta,$$

[1]) The angle AMG indicated as C by Prof. V. D. WAALS has here been called φ, whilst the angle indicated as φ has here been called θ.

giving after some reduction:

$$2\left[(h^2+{}^1/_2\,a^2)\,\theta_1+(2\,a\,h\,\sin\theta_1+{}^1/_2\,a^2\sin\theta_1\cos\theta_1)\right].$$

If we now call the angle CMA, point C lying on the sphere A, $2\,\psi$, we have evidently, θ_1 being $=180-\varphi-2\,\psi$:

$$\left.\begin{aligned}\sin\theta_1 &= \sin(\varphi+2\,\psi)=\sin\varphi\cos 2\,\psi+\cos\varphi\sin 2\,\psi\\ \cos\theta_1 &= -\cos(\varphi+2\,\psi)=-\cos\varphi\cos 2\,\psi+\sin\varphi\sin 2\,\psi\end{aligned}\right\},$$

or as

$$\sin\varphi=\frac{{}^1/_2\,r}{a}\,,\qquad \cos\varphi=\frac{h}{a}\,,\qquad \tan\varphi=\frac{{}^1/_2\,r}{h}$$

$$\sin\psi=\frac{{}^1/_2\,R}{a}\qquad \sin 2\,\psi=\frac{R}{a^2}\sqrt{a^2-{}^1/_4\,R^2}\qquad \cos 2\,\psi=\frac{1}{a^2}(a^2-{}^1/_2\,R^2),$$

also
$$\sin\theta_1=\frac{1}{a^3}\left[{}^1/_2\,r\,(a^2-{}^1/_2\,R^2)+h\,R\sqrt{a^2-{}^1/_4\,R^2}\right]$$

$$\cos\theta_1=\frac{1}{a^3}\left[-h\,(a^2-{}^1/_2\,R^2)+{}^1/_2\,r\,R\sqrt{a^2-{}^1/_4\,R^2}\right];$$

so, taking into consideration the relation

$$h^2=a^2-{}^1/_4\,r^2,$$

after reduction, we find:

$$2ah\sin\theta_1+{}^1/_2\,a^2\sin\theta_1\cos\theta_1=\frac{1}{2\,a^4}\Big[{}^1/_2\,r\sqrt{a^2-{}^1/_4\,r^2}\,(3\,a^4-{}^1/_2\,R^4)+$$

$$+R\sqrt{a^2-{}^1/_4\,R^2}\left(3\,a^4+{}^1/_2\,a^2(R^2-r^2)-{}^1/_4\,R^2r^2\right)\Big]\quad.\quad.\quad(1)$$

$$(h^2+{}^1/_2\,a^2)\,\theta_1=({}^3/_2\,a^2-{}^1/_4\,r^2)\left[\tan^{-1}\frac{-{}^1/_2\,r}{\sqrt{a^2-{}^1/_4\,r^2}}-2\tan^{-1}\frac{{}^1/_2\,R}{\sqrt{a^2-{}^1/_4\,R^2}}\right],(2)$$

θ_1 being equal to $(180-\varphi)-2\,\psi$.

Now the form to be integrated is

$$4\pi\frac{N}{V}\int\Big[(1)+(2)\Big]\times\text{part of segment}\times dh.$$

To simplify still further we put:

$$^1/_2\, r = n\, R \qquad a = y\, R,$$

so that

$$h = R \sqrt{y^2 - n^2}$$

and

$$dh = {}^1/_2\, R \frac{dy^2}{\sqrt{y^2-n^2}} \cdot$$

The above mentioned expression for the part of segment passes into:

$$^1/_3\, R^3 \left[n \sqrt{(1-y^2)(y^2-n^2)} + 2 \tan^{-1} \left(n \sqrt{\frac{1-y^2}{y^2-n^2}} \right) - \right.$$

$$\left. - n\,(3-n^2) \tan^{-1} \sqrt{\frac{1-y^2}{y^2-n^2}} \right] \cdot$$

(1) becomes

$$^1/_2\, R^2 \frac{1}{y^4} \left[n\,(3\,y^4 - {}^1/_2) \sqrt{y^2 - n^2} + \left\{ 3\,y^4 + {}^1/_2\, y^2\,(1 - 4n^2) - n^2 \right\} \sqrt{y^2 - {}^1/_4} \right] .$$

(2) becomes

$$^1/_2\, R^2\,(3\,y^2 - 2\,n^2) \left[\tan^{-1} \frac{-n}{\sqrt{y^2 - n^2}} - 2 \tan^{-1} \frac{^1/_2}{\sqrt{y^2 - {}^1/_4}} \right],$$

so that our integral is now, writing everywhere x for y^2, transformed into:

$$I = {}^1/_3\, \pi\, R^6 \frac{N}{V} \int (A + B + C + D)\, dx,$$

where $\left(\text{paying attention to } dh \text{ being equal to } {}^1/_2\, R \frac{dx}{\sqrt{x - n^2}}\right)$:

$$A = n \sqrt{1-x} \left[\frac{n\,(3\,x^2 - {}^1/_2)}{x^2} \sqrt{x - n^2} + \frac{3\,x^2 + {}^1/_2\, x\,(1-4n^2) - n^2}{x^2} \sqrt{x - {}^1/_4} \right]$$

$$B = n\,(3\,x - 2\,n^2) \sqrt{1 - x} \left[\tan^{-1} \frac{-n}{\sqrt{x - n^2}} - 2 \tan^{-1} \frac{^1/_2}{\sqrt{x - {}^1/_4}} \right]$$

$$C = \left[\frac{n\,(3\,x^2 - {}^1/_2)}{x^2} + \frac{3\,x^2 + {}^1/_2\, x\,(1 - 4\,n^2) - n^2}{x^2} \sqrt{\frac{x - {}^1/_4}{x - n^2}} \right] \times$$

$$\times \left[2 \tan^{-1} \left(n \sqrt{\frac{1-x}{x-n^2}} \right) - n\,(3 - n^2) \tan^{-1} \sqrt{\frac{1-x}{x-n^2}} \right]$$

$$D = \frac{3x - 2n^2}{\sqrt{x - n^2}} \left[\tan^{-1} \frac{-n}{\sqrt{x - n^2}} - 2 \tan^{-1} \frac{1/2}{\sqrt{x - 1/4}} \right] \times$$

$$\times \left[2 \tan^{-1} \left(n \sqrt{\frac{1-x}{x-n^2}} \right) - n(3 - n^2) \tan^{-1} \sqrt{\frac{1-x}{x-n^2}} \right]$$

In the integrations following now we shall for the present not pay attention to the limits for h (or x). To simplify the notation I still propose the following abridgments:

$$\sqrt{\frac{1-x}{x-n^2}} = z \qquad \sqrt{\frac{1-x}{x-1/4}} = z'$$

$$\sqrt{(1-x)(x-n^2)} = p \qquad \sqrt{(1-x)(x-1/4)} = p'$$

We then easily find for the five parts of the integral $\int A\,dx$:

$$\left.\begin{aligned} A_1 &= 3n^2 \int p\,dx = 3/4\, n^2 \left[\Big(2x - (1+n^2)\Big) p - (1-n^2)^2 \tan^{-1} z \right] \\ A_2 &= -1/2\, n^2 \int \frac{p}{x^2}\,dx = 1/2\, n^2 \left[\frac{p}{x} - 2 \tan^{-1} z + \frac{1+n^2}{n} \tan^{-1} nz \right] \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} A_3 &= 3n \int p'\,dx = 3/4\, n \left[(2x - 5/4) p' - 9/16 \tan^{-1} z' \right] \\ A_4 &= 1/2\, n(1-4n^2) \int \frac{p'}{x}\,dx = 1/2\, n (1-4n^2) \left[p' - 5/4 \tan^{-1} z' + \tan^{-1} 1/2\, z' \right] \\ A_5 &= -n^3 \int \frac{p'}{x^2}\,dx = n^3 \left[\frac{p'}{x} - 2 \tan^{-1} z' + 5/2 \tan^{-1} 1/2\, z' \right] \end{aligned}\right\}, \quad (4)$$

as is to be verified by means of the relations.

$$dp = -1/2 \frac{2x - (1+n^2)}{p}\,dx \qquad dp' = -1/2 \frac{2x - 5/4}{p'}\,dx$$

$$d \tan^{-1} z = -1/2 \frac{dx}{p} \qquad d \tan^{-1} z' = -1/2 \frac{dx}{p'}$$

$$d \tan^{-1} nz = -\frac{n}{2} \frac{dx}{xp} \qquad d \tan^{-1} 1/2\, z' = -1/4 \frac{dx}{xp'}$$

Furthermore, as

$$\int (3\,x - 2\,n^2) \sqrt{1-x}\,dx = -\,2\left[(1-{}^2/_3\,n^2)\,(1-x)^{3/2} - {}^3/_5\,(1-x)^{5/2}\right]$$

$$d\,tan^{-1}\frac{-n}{\sqrt{x-n^2}} = \frac{n}{2}\,\frac{dx}{x\sqrt{x-n^2}} \qquad d\,tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}} = -\,{}^1/_4\,\frac{dx}{x\sqrt{x-{}^1/_4}}\,,$$

we find for both parts of $\int B dx$:

$$\left.\begin{aligned}
B_1 &= -\,2\,n\left[(1-{}^2/_3\,n^2)\,(1-x)^{3/2} - {}^3/_5\,(1-x)^{5/2}\right] tan^{-1}\frac{-n}{\sqrt{x-n^2}} - \\
&-n^2\left[\left(\frac{3}{10}\,x - \frac{7}{20} - \frac{13}{60}\,n^2\right)p - \left(\frac{3}{4} - \frac{3}{2}\,n^2 + \frac{13}{60}\,n^4\right) tan^{-1} z + \right.\\
&\qquad\qquad \left. + \frac{4}{n}\left(\frac{1}{5} - \frac{1}{3}\,n^2\right) tan^{-1}\,nz\right] \\
B_2 &= 4\,n\left[(1-{}^2/_3\,n^2)\,(1-x)^{3/2} - {}^3/_5\,(1-x)^{5/2}\right] tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}} - \\
&-n\left[\left(\frac{3}{10}\,x - \frac{19}{80} - \frac{2}{3}\,n^2\right)p' - \left(\frac{27.1}{320} - \frac{11}{6}\,n^2\right) tan^{-1}\,z' + \right.\\
&\qquad\qquad \left. + 8\left(\frac{1}{5} - \frac{1}{3}\,n^2\right) tan^{-1}\,{}^1/_2\,z'\right]
\end{aligned}\right\} \quad (5)$$

The integration of the four parts of $\int C\,dx$ is already more difficult. Comparatively easy are C_1 and C_2:

$$\left.\begin{aligned}
C_1 &= -\,n^2\,(3-n^2)\int\frac{3\,x^2-{}^1/_2}{x^2}\,tan^{-1}\,z\,.\,dx = \\
&= -\,n^2\,(3-n^2)\left[-{}^3/_2\,p + \left\{3x + \frac{1}{2x} - {}^3/_2\,(1+n^2)\right\} tan^{-1} z - \frac{1}{2n}\,tan^{-1} nz\right] \\
C_2 &= 2\,n\int\frac{3\,x^2-{}^1/_2}{x^2}\,tan^{-1}\,nz\,.\,dx = \\
&= 2\,n\left[\frac{p}{4\,n\,x} - 3\,n\,tan^{-1}\,z + \left(3\,x + \frac{1}{2x} - \frac{1+n^2}{4\,n^2}\right) tan^{-1} nz\right]
\end{aligned}\right\} \quad (6$$

To integrate C_3 we add to this integral:

$$-n(3-n^2)\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\,tan^{-1}z\,.\,dx\ ,$$

which afterwards will be subtracted from D_3. So C_3 becomes:

$$C_3' = -n(3-n^2)\int\frac{4x^2+{}^1/_2\,x(1-4n^2)-n^2}{x^2}\sqrt{\frac{x-{}^1/_4}{x-n^2}}\,tan^{-1}z\,.\,dx\ .$$

But now we find:

$$\frac{4x^2+{}^1/_2\,x(1-4n^2)-n^2}{x^2}\sqrt{\frac{x-{}^1/_4}{x-n^2}}\,dx = d\left[\frac{4(x-{}^1/_4)}{x}\sqrt{(x-{}^1/_4)(x-n^2)}\right],$$

so that we get (the various developments — as was said before — will be published elsewhere):

$$C_3' = -n(3-n^2)\left[\frac{4(x-{}^1/_4)}{x}\sqrt{(x-{}^1/_4)(x-n^2)}\,tan^{-1}z\ -\right.$$
$$\left.-\,2\left\{p'+{}^1/_4\,tan^{-1}z'+{}^1/_4\,tan^{-1}\,{}^1/_2\,z'\right\}\right].\quad(7)$$

Likewise to the integral C_4 is added:

$$2\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\,tan^{-1}nz\,.\,dx\ ,$$

which shall directly be subtracted from D_4. For

$$C_4' = 2\int\frac{4x^2+{}^1/_2\,x(1-4n^2)-n^2}{x^2}\sqrt{\frac{x-{}^1/_4}{x-n^2}}\,tan^{-1}nz\,.\,dx$$

we find in quite the same way as for C_3':

$$C_4' = 2\left[\frac{4(x-{}^1/_4)}{x}\sqrt{(x-{}^1/_4)(x-n^2)}\,tan^{-1}nz\ +\right.$$
$$\left.+\,2n\left\{\frac{1}{4}\,\frac{p'}{x}-2\,tan^{-1}z'+\frac{11}{8}\,tan^{-1}\,{}^1/_2\,z'\right\}\right].\quad(8)$$

As $\int \frac{3x-2n^2}{\sqrt{x-n^2}}dx$ is equal to $2x\sqrt{x-n^2}$, we shall find after various reductions successively for the 4 parts of $\int D\,dx$:

$$D_1 = -n(3-n^2)\int \frac{3x-2n^2}{\sqrt{x-n^2}}\tan^{-1}\frac{-n}{\sqrt{x-n^2}}\tan^{-1}z\,.\,dx =$$

$$= -n(3-n^2)\Big[2x\sqrt{x-n^2}\tan^{-1}\frac{-n}{\sqrt{x-n^2}}\tan^{-1}z -$$

$$-{}^2/_3(x+2)\sqrt{1-x}\tan^{-1}\frac{-n}{\sqrt{x-n^2}}+{}^5/_6np-n(x-{}^3/_2-{}^5/_6n^2)\tan^{-1}z-{}^4/_3\tan^{-1}nz\Big].(9)$$

Likewise:

$$D_2 = 2\int \frac{3x-2n^2}{\sqrt{x-n^2}}\tan^{-1}\frac{-n}{\sqrt{x-n^2}}\tan^{-1}nz\,.\,dx =$$

$$= 2\Big[2x\sqrt{x-n^2}\tan^{-1}\frac{-n}{\sqrt{x-n^2}}\tan^{-1}nz - 2n\sqrt{1-x}\tan^{-1}\frac{-n}{\sqrt{x-n^2}} +$$

$$+ 3n^2\tan^{-1}z - n(x+2)\tan^{-1}nz\Big] \quad . \quad . \quad . \quad . \quad (10)$$

The integral D_3, viz.

$$D_3 = 2n(3-n^2)\int \frac{3x-2n^2}{\sqrt{x-n^2}}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\tan^{-1}z\,.\,dx$$

can first be reduced to

$$D_3 = 2n(3-n^2)\Big[2x\sqrt{x-n^2}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\tan^{-1}z +$$

$$+ {}^1/_2\int\sqrt{\frac{x-n^2}{x-{}^1/_4}}\tan^{-1}z\,.\,dx + \int\frac{x}{\sqrt{1-x}}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\,.\,dx\,.$$

But as $\int\sqrt{\frac{x-n^2}{x-{}^1/_4}}\tan^{-1}z\,.\,dx$ can be transformed into

$$-\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\tan^{-1}z\,.\,dx + 2\sqrt{(x-{}^1/_4)(x-n^2)}\tan^{-1}z - p' - {}^3/_4\tan^{-1}z',$$

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we subtract from D_3 the part $2\,n(3-n^2)\times -{}^1/_2\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\tan^{-1} z\,.\,dx$, which part, as we saw above, was already added to C_3. After various lengthy developments we find at last:

$$D_3' = 2n\,(3-n^2)\Big[2\,x\sqrt{x-n^2}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\tan^{-1} z -$$
$$-\,{}^2/_3\,(x+2)\sqrt{1-x}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}} + \sqrt{(x-{}^1/_4)(x-n^2)}\tan^{-1} z -$$
$$-\frac{2}{3}p' - \frac{11}{12}\tan^{-1} z' + \frac{4}{3}\tan^{-1}{}^1/_2\,z'\Big] \quad . \quad . \quad . \quad (11)$$

And in the same way for

$$D_4 = -4\int\frac{3\,x-2\,n^2}{\sqrt{x-n^2}}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\tan^{-1} nz\,.\,dx,$$

after transformation of the integral ${}^1/_2\int\sqrt{\frac{x-n^2}{x-{}^1/_4}}\tan^{-1} nz\,.\,dx$ into

$${}^1/_2\Big[-\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\tan^{-1} nz\,.\,dx + 2\,\sqrt{(x-{}^1/_4)(x-n^2)}\tan^{-1} nz -$$
$$-\,2\,n\tan^{-1} z' + n\tan^{-1}{}^1/_2\,z'\Big],$$

and subtraction of the integral already added to C_4:

$$2\int\sqrt{\frac{x-{}^1/_4}{x-n^2}}\tan^{-1} nz\,.\,dx\,,$$

we find:

$$D_4' = -4\Big[2\,x\sqrt{x-n^2}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\tan^{-1} nz - 2n\sqrt{1-x}\tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}} +$$
$$+\sqrt{(x-{}^1/_4)(x-n^2)}\tan^{-1} nz - 2n\tan^{-1} z' + {}^5/_2\,n\tan^{-1}{}^1/_2 z'\Big]\,.\,(12)$$

If we now join all the similar terms, we shall find for

$$\int(A+B+C+D)\,dx$$

the following:

$$
\left.
\begin{array}{l}
\left[\frac{1+n^2}{2\,x} + {}^6/_5\, n^2 x + {}^2/_5\, n^2 (4-3\,n^2)\right] \sqrt{(1-x)(x-n^2)} + \\
\quad + \left[\frac{n\,(1+n^2)}{x} + {}^6/_5\, n\, x + {}^1/_5\, n\,(9-10\,n^2)\right] \sqrt{(1-x)(x-{}^1/_4)} + \\
+ \left[- n^2 (3-n^2)\left(2\,x + \frac{1}{2x}\right) - n^2 (1-2\,n^2 + {}^6/_5\, n^4) + \right. \\
\quad \left. + n\,(3-n^2)\left(\frac{1}{x} - 2\right)\sqrt{(x-{}^1/_4)(x-n^2)}\right] tan^{-1}\sqrt{\frac{1-x}{x-n^2}} + \\
+ \left[2\,n\left(2\,x+\frac{1}{2x}\right) - \frac{1}{n}({}^1/_2 - {}^7/_{10}\, n^2) - \right. \\
\qquad \left. - 2\left(\frac{1}{x} - 2\right)\sqrt{(x-{}^1/_4)(x-n^2)}\right] tan^{-1}\left(n\sqrt{\frac{1-x}{x-n^2}}\right) - \\
- \frac{21}{5}\, n\, tan^{-1}\sqrt{\frac{1-x}{x-{}^1/_4}} + \frac{39}{10}\, n\, tan^{-1}\, {}^1/_2 \sqrt{\frac{1-x}{x-{}^1/_4}} + \\
+ 2n\sqrt{1-x}\,(-{}^2/_5 + ({}^4/_5 - n^2)\, x + {}^3/_5\, x^2)\left(tan^{-1}\frac{-n}{\sqrt{x-n^2}} - 2\, tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\right) \\
+ 2\,x\sqrt{x-n^2}\left(tan^{-1}\frac{-n}{\sqrt{x-n^2}} - 2\, tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}\right) \times \\
\quad \times \left[2\, tan^{-1}\left(n\sqrt{\frac{1-x}{x-n^2}}\right) - n\,(3-n^2)\, tan^{-1}\sqrt{\frac{1-x}{x-n^2}}\right]
\end{array}
\right\} \quad (13)
$$

Let us now introduce the limits for h. These are (see the paper of Prof. v. D. WAALS) for values of r lying between R and $R\sqrt{3}$:

$$\frac{R^2 - {}^1/_2\, r^2}{2\sqrt{R^2 - {}^1/_4\, r^2}} \quad \text{and} \quad \sqrt{R^2 - {}^1/_4\, r^2}\ ,$$

whilst for values of r between $R\sqrt{3}$ and $2\,R$ they are:

$$-\sqrt{R^2 - {}^1/_4} \quad \text{and} \quad \sqrt{R^2 - {}^1/_4\, r^2}\ .$$

So with our notations we get ($h = R\sqrt{x-n^2}$):

For $n = {}^1/_2$ to ${}^1/_2\sqrt{3}$:

$$\sqrt{x-n^2} \text{ from } \frac{1-2n^2}{2\sqrt{1-n^2}} \quad \text{to} \quad \sqrt{1-n^2}\left(x \text{ from } \frac{1}{4(1-n^2)} \quad \text{to} \quad 1\right);$$

For $n = {}^1/_2\sqrt{3}$ to 1:

$$\sqrt{x-n^2} \text{ from } -\sqrt{1-n^2} \quad \text{to} \quad \sqrt{1-n^2} \quad (x \text{ from } 1 \text{ to } 1)\,.$$

Substitution of these limits in (13) gives, paying attention to the circumstance that wherever $\sqrt{x-n^2}$ appears in tan^{-1}, the value of tan^{-1} is equal to π for $\sqrt{x-n^2} = -\sqrt{1-n^2}$, when at the same time $\sqrt{1-x}$ (which becomes 0) appears in the numerator:

$$\left.\begin{aligned}
I_a(n = {}^1/_2 \text{ to } {}^1/_2\sqrt{3}) &= -({}^1/_2 + {}^3/_2\,n^2 - {}^3/_5\,n^4)\sqrt{3-4n^2} + \\
&+ n^2(7-4n^2+{}^6/_5\,n^4)\,tan^{-1}\frac{\sqrt{3-4n^2}}{1-2n^2} + \frac{1}{n}\left(\frac{1}{2}-\frac{47}{10}n^2\right)tan^{-1}\frac{n\sqrt{3-4n^2}}{1-2n^2} + \\
&+ \frac{21}{5}\,n\,tan^{-1}\frac{\sqrt{3-4n^2}}{n} - \frac{39}{10}\,n\,tan^{-1}\frac{\sqrt{3-4n^2}}{2n} \\
I_b(n = {}^1/_2\sqrt{3} \text{ to } 1) &= \pi\left[\frac{1}{2n} - \frac{57}{10}n + \frac{17}{2}n^2 - \frac{9}{2}n^4 + \frac{6}{5}n^6\right] + \\
&+ {}^1/_2\,\pi\sqrt{3}\,(2-3n+n^3)\sqrt{1-n^2} + \\
&+ 2\pi(2-3n+n^3)\sqrt{1-n^2}\left(tan^{-1}\frac{n}{\sqrt{1-n^2}} - {}^1/_3\,\pi\right)
\end{aligned}\right\} \quad (14)$$

Here I have also availed myself of the circumstance that

$$tan^{-1}\frac{-n}{\sqrt{x-n^2}} - 2\,tan^{-1}\frac{{}^1/_2}{\sqrt{x-{}^1/_4}}$$

disappears for $x = \dfrac{1}{4(1-n^2)}\left(\sqrt{x-n^2} = \dfrac{1-2n^2}{2\sqrt{1-n^2}}\right)$.

The expressions found for I_a and I_b have been verified by me in various ways and every time found true. They are both still to be multiplied by ${}^1/_3\,\pi R^6 \dfrac{N}{V}$. Before passing to the second integration with respect to n, we must calculate a *complementary term* for the cases (not mentioned in the cited paper) in which point M falls outside the segment and an *entire* segment is included by a third sphere. It is easy to see that that complementary term is obtained out of

$$I'_b = 2\int\int \frac{N}{V}\, 2\,\pi\,(h + a\cos O)^2\, dh\, dO \times \text{segment},$$

which produces after some reductions,

Segment being equal to $\frac{1}{3}\pi\,(2\,R^3 - \frac{3}{2}R^2 r + \frac{1}{8}r^3) = \frac{1}{3}\pi\,R^3\,(2 - 3\,n + n^3)$:

$$I'_b = \tfrac{1}{3}\,\pi^2\,R^6\,\frac{N}{V}\,(2 - 3\,n + n^3)\int\Bigg[\frac{n\,(3\,x^2 - \frac{1}{2})}{x^2} +$$

$$+\frac{3x^2 + \frac{1}{2}x(1 - 4n^2) - n^2}{x^2}\sqrt{\frac{x - \frac{1}{4}}{x - n^2}} + \frac{3x - 2n^2}{\sqrt{x - n^2}}\left(\tan^{-1}\frac{-n}{\sqrt{x - n^2}} - 2\tan^{-1}\frac{\frac{1}{2}}{\sqrt{x - \frac{1}{4}}}\right)\Bigg]\,dx.$$

Of this integral we mention only the result taken between the limits $\sqrt{x - n^2} = \dfrac{1 - 2\,n^2}{2\sqrt{1 - n^2}}$ to

$$\sqrt{x - n^2} = -\sqrt{1 - n^2}\ \left(x \text{ from } \frac{1}{4\,(1 - n^2)} \text{ to } 1\right).$$

It is evident that this integral relates only to the second tempo ($n = \frac{1}{2}\sqrt{3}$ to 1) .

We find:

$$I'_b = \tfrac{1}{3}\,\pi^2\,R^6\,\frac{N}{V}\,(2 - 3\,n + n^3)\,\Big[\tfrac{1}{2}\,n - \tfrac{1}{2}\sqrt{3\,(1 - n^2)} -$$

$$-2\sqrt{1 - n^2}\left(\tan^{-1}\frac{n}{\sqrt{1 - n^2}} - \tfrac{1}{3}\,\pi\right)\Big], \quad . \quad . \quad (15)$$

by which most remarkably the above named value of I_b is considerably simplified and, with the omission of the factor $\frac{1}{3}\pi\,R^6\,\frac{N}{V}$, passes into:

$$I_b + I''_b = \pi\left[\frac{1}{2n} - \frac{47}{10}\,n + 7\,n^2 - 4\,n^4 + \frac{6}{5}\,n^6\right] \quad . \quad . \quad (16)$$

If we now multiply by

$$4\,\pi\,r^2\,dr \times \frac{N}{V} = 32\,\pi\,R^3\,\frac{N}{V} \times n^2\,dn,$$

we have still to integrate:

$$I = \frac{32}{3}\,\pi^2\,R^9\,\frac{N^2}{V^2}\left[\int_{1/_2}^{1/_2\sqrt{3}} I_a\,n^2\,dn + \int_{1/_2\sqrt{3}}^{1} (I_b + I''_b)\,n^2\,dn\right]$$

This integration we break up into parts again.

$$I_1 = \int_{1/_2}^{1/_2\sqrt{3}} (-\,{}^1/_2\,n^2 - {}^3/_2\,n^4 + {}^3/_5\,n^6)\,\sqrt{3 - 4\,n^2}\,dn$$

furnishes:

$$I_1 = \left[\left(\frac{735}{8192}\,n - \frac{89}{1024}\,n^3 - \frac{83}{320}\,n^5 + \frac{3}{40}\,n^7\right)\sqrt{3 - 4\,n^2} + \right.$$

$$\left. + \frac{2205}{16384}\,tan^{-1}\,\frac{\sqrt{3 - 4\,n^2}}{2\,n}\right]_{1/_2}^{1/_2\sqrt{3}},$$

as is to be verified by means of the relations:

$$d\sqrt{3 - 4\,n^2} = \frac{-\,4\,n\,dn}{\sqrt{3 - 4\,n^2}} \qquad d\,tan^{-1}\,\frac{\sqrt{3 - 4\,n^2}}{2\,n} = \frac{-\,2\,dn}{\sqrt{3 - 4\,n^2}}.$$

Introduction of the limits gives further:

$$I_1 = -\,\frac{2169}{5 \times 16384}\,\sqrt{2} - \frac{2205}{16384}\,tan^{-1}\,\sqrt{2}\,. \quad . \quad . \quad (17)$$

In the second place, see (14) and (16):

$$I_2 = \int_{1/_2}^{1/_2\sqrt{3}} (7\,n^4 - 4\,n^6 + {}^6/_5\,n^8)\,tan^{-1}\,\frac{\sqrt{3 - 4\,n^2}}{1 - 2\,n^2}\,.\,dn + \int_{1/_2\sqrt{3}}^{1} (7\,n^4 - 4\,n^6 + {}^6/_5\,n^8)\pi\,.$$

As $\tan^{-1} \frac{\sqrt{3-4n^2}}{1-2n^2}$ takes for $n = \frac{1}{2}\sqrt{3}$ the value π, we find by means of

$$d \tan^{-1} \frac{\sqrt{3-4n^2}}{1-2n^2} = \frac{2n}{1-n^2} \frac{dn}{\sqrt{3-4n^2}},$$

and, paying attention to the relation $\tan^{-1} 2\sqrt{2} = \pi - 2\tan^{-1}\sqrt{2}$:

$$I_2 = \left(\frac{5719}{15.256} - \frac{127}{7.32}\right)\pi + \left(\frac{169}{15.128} - \frac{1}{7.16}\right)\tan^{-1}\sqrt{2} -$$

$$- \int_{1/2}^{1/2\sqrt{3}} \frac{14/5\, n^6 - 8/7\, n^8 + 4/15\, n^{10}}{(1-n^2)\sqrt{3-4n^2}} dn \quad . \quad . \quad (18)$$

In the third place:

$$I_3 = \int_{1/2}^{1/2\sqrt{3}} \left(\frac{1}{2} n - \frac{47}{10} n^3\right) \tan^{-1} \frac{n\sqrt{3-4n^2}}{1-2n^2} . dn + \int_{1/2\sqrt{3}}^{1} \left(\frac{1}{2} n - \frac{47}{10} n^3\right) \pi\, dn .$$

With $d \tan^{-1} \frac{n\sqrt{3-4n^2}}{1-2n^2} = \frac{3-2n^2}{1-n^2} \frac{dn}{\sqrt{3-4n^2}}$ we easily find:

$$I_3 = -\frac{37}{40}\pi + \frac{7}{640}\tan^{-1}\sqrt{2} - \int_{1/2}^{1/2\sqrt{3}} \frac{\frac{3}{4}n^2 - \frac{161}{40}n^4 + \frac{47}{20}n^6}{(1-n^2)\sqrt{3-4n^2})} dn \quad . \quad (19)$$

For

$$I_4 = \int_{1/2}^{1/2\sqrt{3}} \left[\frac{21}{5} n^3 \tan^{-1} \frac{\sqrt{3-4n^2}}{n} - \frac{39}{10} n^3 \tan^{-1} \frac{\sqrt{3-4n^2}}{2n}\right] dn$$

we find by means of

$$d \tan^{-1} \frac{\sqrt{3-4n^2}}{n} = \frac{-dn}{(1-n^2)\sqrt{3-4n^2}}, \quad d \tan^{-1} \frac{\sqrt{3-4n^2}}{2n} = \frac{-2\,dn}{\sqrt{3-4n^2}}:$$

$$I_4 = \left(-\frac{21}{320}\pi + \frac{21}{160}\tan^{-1}\sqrt{2}\right) + \frac{39}{640}\tan^{-1}\sqrt{2} +$$

$$+\int_{1/2}^{1/2\sqrt{3}} \frac{\frac{21}{20}n^4 - \frac{39}{20}n^4(1-n^2)}{(1-n^2)\sqrt{3-4n^2}}dn,$$

where again in the first part I made use of the relation $\tan^{-1} 2\sqrt{2} = \pi - 2\tan^{-1}\sqrt{2}$.

If we then join the terms obtained, we shall get besides the foremost factor

$$\left.\begin{aligned} I = &-\frac{2169}{5.16384}\sqrt{2} + \left(\frac{383}{3.256} - \frac{127}{7.32}\right)\pi + \\ &+\left(-\frac{2205}{16384} + \frac{559}{15.128} - \frac{1}{7.16}\right)\tan^{-1}\sqrt{2} + \\ &+\int_{1/2}^{1/2\sqrt{3}} \frac{-\,^3/_4\,n^2 + \,^{25}/_8\,n^4 - \,^{16}/_5\,n^6 + \,^8/_7\,n^8 - \,^4/_{15}\,n^{10}}{(1-n^2)\sqrt{3-4n^2}}\,dn \end{aligned}\right\} \quad (20)$$

For the integrating of this last integral we again refer to the more lengthy paper; it is sufficient to mention the result for I. I only draw attention to the fact that after having successively determined

$$\int \frac{n^{2k}\,dn}{\sqrt{3-4n^2}},$$

where $k = 1, 2, 3, 4$, besides

$$\int \frac{n^2\,dn}{(1-n^2\sqrt{3-4n^2}} = \pi - \,^5/_2\tan^{-1}\sqrt{2},$$

all the above-named integrals are found by parts. So is e.g.

$$\int \frac{n^4\,dn}{(1-n^2)\sqrt{3-4n^2}} = \int \frac{n^2\,dn}{(1-n^2)\sqrt{3-4n^2}} - \int \frac{n^2\,dn}{\sqrt{3-4n^2}},$$

etc. The result now becomes, after multiplication by the foremost factor and by N for all N spheres:

$$\frac{32}{3}\pi^2 R^9 \frac{N^3}{V^2}\left[\frac{73}{7.45.64}\sqrt{2} - \frac{153}{7.5.256}\pi + \frac{153}{7.5.64}\tan^{-1}\sqrt{2}\right] \quad . (21)$$

If for a moment we call the expression between brackets ω, this may be written:

$$\left(\frac{4}{3}\pi R^3\right)^3 \frac{N^3}{V^2} \times \frac{9}{2}\frac{\omega}{\pi}.$$

For the double volume of the N spheres of distance remains also, after paying attention to the 1st and 2nd corrections:

$$N \, . \, {}^4/_3\, \pi R^3 - ({}^4/_3\, \pi R^3)^2 \frac{17}{64}\frac{N^2}{V} + ({}^4/_3\, \pi R^3)^3 . \frac{9}{2}\frac{\omega}{\pi}\frac{N^3}{V^2} =$$

$$= 8\,b\left[1 - \frac{17}{64}\frac{8\,b}{V} + \frac{9}{2}\frac{\omega}{\pi}\frac{64\,b^2}{V^2}\right],$$

$N \, . \, {}^4/_3\, \pi R^3$ being equal to $8\,b$. If now $4\,b = b_\infty$, then in

$$2\,b_\infty\left[1 - \frac{17}{32}\frac{b_\infty}{V} + 18\frac{\omega}{\pi}\left(\frac{b_\infty}{V}\right)^2\right]$$

the 2nd correction sought for evidently becomes equal to $18\,\frac{\omega}{\pi}$, so:

$$\beta = \frac{9}{35.32\,\pi}\left[\frac{73}{9}\sqrt{2} + 153\left(\tan^{-1}\sqrt{2} - {}^1/_4\,\pi\right)\right],$$

or

$$\beta = \frac{1}{1120\,\pi}\left[73\sqrt{2} + 81.17\left(\tan^{-1}\sqrt{2} - {}^1/_4\,\pi\right)\right] \quad , (22)$$

this being our definite result. The value of this is, exact in 4 decimals,

$$\underline{\beta = 0{,}0958,}$$

so almost $^1/_{10}$, whereas the 1st correction was fully $^1/_2$.