sentation as the one used here, a horizontal line connecting A and B-as was the opinion of KIPPING and POPE, but may quite as well show a maximum or a minimum, which then however lies at 50 pCt. I even suspect such to be the case in camphersulfonic chlorid and carvontribromide.

According to these views, neither a higher nor a lower melting-point furnishes a proof on the nature of an inactive substance, but the study of the entire melting-line does.

A single curve serves in case of mixed crystals, two curves in case of an inactive conglomerate, three in case of a combination.

Other remarkable phenomena may still present themselves, in case transformations of the combination, mixture or conglomerate appear after the congelation.

Mathematics. — "*A geometrical interpretation of the invariant* $\underset{n+1}{\Pi}(ab)^2$ *of a binary form* a^{2n} *of even degree*". By Prof. P. H. SCHOUTE.

With regard to the creation of the beautiful theory of the invariants undoubtedly very much is due to SYLVESTER as well as to ARONHOLD, BOOLE, BRIOSCHI, CAYLEY, CLEBSCH, GORDAN, HERMITE and others. As early as 1851, indeed, he developed in his treatise: "*On a remarkable discovery in the theory of canonical forms and of hyperdeterminants*" (*Phil. Mag.*, Vol. II of Series 4, p. 391—410) the foundation upon which the theory of the canonical forms is based. The principal contents consist of the proof of two theorems. According to the first the general binary form of the odd degree $2n-1$ can always be written in a single way as the sum of the $2n-1^{st}$ powers of n binary linear forms; according to the second the binary form of the even degree $2n$ can be written as the sum of the $2n^{th}$ powers of n binary linear forms — and in that case in a single way too — only when a certain invariant vanishes. For this invariant with which we shall deal here compare a.o. GUNDELFINGER's treatise in the "*Journ. f. Math.*", Vol. 100, p. 413—424, 1883, and SALMON's "*Modern higher algebra*", 4th ed., p. 156, 1885. So the theory of invariants of a certain form of any kind is ruled by the question about the minimum number of homonymous powers of linear forms by which it can be represented. (Compare a.o. REYE in the "*Journ. f. Math.*", Vol. 73rd, p. 114—122). With this the theory of involutions of a higher dimension and order are closely allied. Likewise theorems are deduced from it relating to hypergeometry. For all this we refer to two important papers by Mr. W. FR. MEYER. The first, published at Tübingen 1883, is entitled; "*Apolarität und Rationale Curven*"; the second, inserted as "*Bericht über den gegenwärtigen Stand der Invariantentheorie*" in the 1st Vol. published in 1892 of the "*Jahresbericht der Deutschen Mathematiker-Vereinigung*", is an invaluable report about this branch of Mathematics. On page 365 of the former work a theorem appears under γ_4 which is closely connected with our subject.

It goes without saying that it must be possible to reach conversely the above quoted theorems of SYLVESTER and the higher involutions connected with them starting from the theory of polydimensional space. Indeed, Mr. CLIFFORD has stated already in 1878 in his important treatise "*On the classification of loci*" (*Phil. Trans.*, Vol. 169, part 2nd, p. 663—681) that in this direction a geometrical interpretation of any invariant of a binary form is to be found. So in trying to determine a certain locus in space with an even number of dimensions I have fallen back upon a geometrical interpretation of the invariant of SYLVESTER; however examining the above mentioned literature I soon discovered that this interpretation had already been found.

Yet I wish to publish my study. In the first place because it may prove that the geometrical way is at least equally simple as the algebraical. Secondly on account of its containing a method of elimination I have as yet nowhere met with in this form. Thirdly because it is not quite impossible that entirely corresponding investigations may lead to a geometrical interpretation of other general invariants [1]).

1. A curve allows of a twofold infinite number of chords, containing together a threefold infinite number of points. If this curve is situated in the space S^3 with three dimensions these points will fill the whole space and one or more of these chords will pass through any given point. If the curve is situated in the space S^4 with four dimensions the locus of the points through which chords pass, i.e. the locus of those chords themselves, is a curved space of the third order. The point from which we start here is the investigation of this curved space for the simplest possible case, namely

[1]) I think e.g. of the invariant $(ab)^{2n}(cd)^{2n}(ac)(bd)$ of a general binary form a_x^{2n+1} of odd degree (compare SALMON l. c., p. 129, problem 2nd) forming an extension of the discriminant of a_x^3, of which as yet no general algebraical interpretation seems known.

that of the normal curve of the space S^4 that is of the rational C_4 of the fourth order, represented by the equations:

$$\frac{x_0}{1} = \frac{x_1}{\lambda} = \frac{x_2}{\lambda^2} = \frac{x_3}{\lambda^3} = \frac{x_4}{\lambda^4},$$

where x_0, x_1, x_2, x_3, x_4 are the homogeneous coordinates of the point L of the curve belonging to the parametervalue λ.

2. If λ_1 and λ_2 are the parametervalues of any two points L_1 and L_2 on C_4, the coordinates of any point A of the line joining L_1 and L_2 are given by the relations

$$\left.\begin{aligned} x_0 &= p_1 + p_2 \\ x_1 &= p_1\lambda_1 + p_2\lambda_2 \\ x_2 &= p_1\lambda_1^2 + p_2\lambda_2^2 \\ x_3 &= p_1\lambda_1^3 + p_2\lambda_2^3 \\ x_4 &= p_1\lambda_1^4 + p_2\lambda_2^4 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

and now by eliminating the four quantities λ_1, λ_2, p_1, p_2 we find the equation of the locus required.

The result of this elimination is the cubic curved space

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0 \quad \ldots\ldots\ldots \quad (2)$$

For if

$$\begin{vmatrix} p_1 + p_2 & p_1\lambda_1 + p_2\lambda_2 & p_1\lambda_1^2 + p_2\lambda_2^2 \\ p_1\lambda_1 + p_2\lambda_2 & p_1\lambda_1^2 + p_2\lambda_2^2 & p_1\lambda_1^3 + p_2\lambda_2^3 \\ p_1\lambda_1^2 + p_2\lambda_2^2 & p_1\lambda_1^3 + p_2\lambda_2^3 & p_1\lambda_1^4 + p_2\lambda_2^4 \end{vmatrix}$$

is written down, it is immediately evident that every combination of partial columns vanishes after easy simplifications, two of the three columns being equal to each other [1]).

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[1]) As I already stated above, I nowhere met with this simple deduction of the invariant of SYLVESTER which can be pursued through all spaces S^{2n}. In the original

3. The osculating space belonging to the point L of C_4 has for equation

$$x_0 \lambda^4 - 4 x_1 \lambda^3 + 6 x_2 \lambda^2 - 4 x_3 \lambda + x_4 = 0 \quad . \; . \; . \quad (3)$$

So the coordinates x_0, x_1, x_2, x_3, x_4 are the coefficients of the form

$$x_0 (-\lambda)^4 + 4 x_1 (-\lambda)^3 + 6 x_2 (-\lambda)^2 + 4 x_3 (-\lambda) + x_4$$

of the fourth order in $(-\lambda)$, bare of the binomial factors.

So the result (2) can be written as $j = 0$ and represented symbolically by $(bc)^2 (ca)^2 (ab)^2 = 0$ (see a. o. CLEBSCH-LINDEMANN's "*Vorlesungen über Geometrie*", I, page 229). From this ensues at the same time that any point of the obtained locus is distinguished moreover from any point taken at random in the space S^4 by the property that the four osculating spaces of C_4 passing through it belong to four harmonic points of the curve. We shall point this out more directly. We suppose in the formulae (1) the quantities $\lambda_1, \lambda_2, p_1, p_2$ to be given; then by substituting in (3) the values ensuing from this for x_k we shall find

$$p_1 (\lambda - \lambda_1)^4 + p_2 (\lambda - \lambda_2)^4 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

as the equation which gives us the parametervalues of the four points L of C_4, whose osculating spaces intersect in the point A of the line $L_1 L_2$ given by (1). If for convenience' sake we take the points L_1 and L_2 as base points with the parametervalues 0 and ∞, this equation can be reduced to $\lambda^4 - 1 = 0$ and the roots $1, -1, \sqrt{-1}, -\sqrt{-1}$ show immediately that the pairs of points belonging to $(1, -1)$ and $(\sqrt{-1}, -\sqrt{-1})$ separate each other harmonically, whilst each of these pairs behaves in the same way with reference to the pair of base points $L_1 L_2$ belonging to $(0, \infty)$. By this, not only the harmonic position of the four points (4) has been indicated but moreover the following theorem has been proved:

"Any two points L_1, L_2 on C_4 determine on this curve a quadratic involution I_2 of which they are the double points. If of "this I_2 we join two pairs separating each other harmonically we "get the biquadratic involution I_4 represented by the equation (4)

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work (*Phil. Mag.* l. c.) first a binary form of odd degree is discussed and the invariant belonging to forms of even degree are reached at the end by a round-about way. In SALMON (l. c.) and otherwhere the determinant is regarded as an extension of the covariant of HESSE built up out of second differentialquotients, here differentialquotients of the 4th order, etc.

"characterized by the particularity that each of the two points "L_1, L_2 counted four times represents a quadruple of it. The osculating spaces belonging to the points of any quadruple intersect in "a point A of $L_1 L_2$. And if this quadruple describes I_4, the point "A generates on $L_1 L_2$ a series of points in projective correspondence "with I_4."

Moreover we easily find [1]):

"If A and A' are two points of the line $L_1 L_2$ harmonically separated by L_1 and L_2, the quadruple of I_4 belonging to A has the "combination of L_1 and L_2 with the quadruple of I_4 belonging to "A' for its sextic covariant T. And combination of the quadruples "belonging to A and A' generates an involution of the eighth order."

The indicated I_8 is represented by the equation

$$p_1^2 (\lambda - \lambda_1)^8 - p_2^2 (\lambda - \lambda_2)^8 = 0 \, ;$$

so it is characterized by the particularity that each of the two points L_1 and L_2 counted eight times represents an octuple of it.

4. We now pass to the space S^6 and there we determine the locus of the planes having three points L_1 L_2 L_3 in common with the normal curve C_6

$$\frac{x_0}{1} = \frac{x_1}{\lambda} = \ldots = \frac{x_5}{\lambda^5} = \frac{x_6}{\lambda^6}$$

of that space. This is obtained by eliminating the six quantities λ_1, λ_2, λ_3, p_1, p_2, p_3 between the seven relations

$$x_k = p_1 \lambda_1^k + p_2 \lambda_2^k + p_3 \lambda_3^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where k must take the values 0, 1, ... 5, 6. In quite the same way as above we find here the curved space S_4^5 with five dimensions of the fourth order represented by

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \\ x_3 & x_4 & x_5 & x_6 \end{vmatrix} = 0.$$

[1]) In fact, the form T of $x_1^4 \pm x_2^4$ is $x_1 x_2 (x_1^4 \mp x_2^4)$.

The first member of this equation is again an invariant of the sextic

$$x_0(-\lambda)^6 + 6\,x_1(-\lambda)^5 + 15\,x_2(-\lambda)^4 + \ldots + 6\,x_5(-\lambda) + x_6,$$

which made equal to nought indicates the osculating space belonging to the point L.

Now we find in the ordinary manner by passing to the use of symbolic coefficients and by noticing their mutual equivalence that the indicated invariant may be represented by $(ab)^2(ac)^2(ad^2)(bc)^2(bd)^2(cd)^2$. Naturally if the invariant [1]) vanishes there is a connection between the six points of C_6 whose osculating spaces pass through the point A of the plane $L_1\,L_2\,L_3$ indicated by the formulae (5), for substitution of the values following out of (5) for the seven coordinates x_k in the equation of the osculating space of the point L gives

$$p_1(\lambda-\lambda_1)^6 + p_2(\lambda-\lambda_2)^6 + p_3(\lambda-\lambda_3)^6 = 0 \quad . \quad . \quad . \quad (6)$$

So we have the following theorem:

"Any three points L_1, L_2, L_3 on C_6 determine on this curve an "involution I_6^2 of the second dimension and the sixth order of which "each of the three points counted six times represents a sextuple. "The osculating spaces belonging to the points of any sextuple inter- "sect in a point A of the plane $L_1\,L_2\,L_3$; if the sextuple describes "I_6^2, the point A generates in the plane $L_1\,L_2\,L_3$ a plane system in "projective correspondence with I_6^2."

The considered invariant of a_x^6 is indicated by SYLVESTER as "catalecticant" because its vanishing is the condition under which a_x^6 can be represented as the sum of three sextic powers; in connection with this an a_x^6 allowing this reduction is called a "meiocatalectic" sextic (*Phil. Mag.* l. c. page 408).

5. Finally we examine in the space S^{2n} the locus of the linear space S^{n-1} having n points $L_1, L_2, \ldots L_n$ in common with the normal curve C_{2n} represented by

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[1]) If according to the common notation $f \equiv a_x^6$ and $k \equiv (f, f)^4 = (ab)^4 a_x^2 b_x^2 =$ the fourth transvectant of f with itself, then the indicated invariant is the fourth transvectant $(k, k)^4$ of k with itself (see GORDAN-KIRSCHENSTEINER "*Vorlesungen über Invariantentheorie*", Vol 2, page 286).

For the following case $f \equiv a_x^8$ we have got to deal with an invariant of the fifth order in the coefficients. There being (see a.o. VON GALL's two papers in the "*Math. Ann.*" Vol 17, p. 31—51 and 139—152, 1890 on "*Das vollständige Formensystem einer binären Form achter Ordnung*") but *one* invariant of this kind, our invariant must in this case correspond to the one indicated by the sign $f_{k,k}$.

$$\frac{x_0}{1} = \frac{x_1}{\lambda} = \dots\dots = \frac{x_{2n-1}}{\lambda^{2n-1}} = \frac{x_{2n}}{\lambda^{2n}}.$$

For this the $2n$ quantities $\lambda_1, \lambda_2, \dots \lambda_n, p_1, p_2, \dots p_n$ must be eliminated between the $2n+1$ equations

$$x_k = \sum_{l=1}^{l=n} p_l \lambda_l^k, \quad \dots\dots\dots\dots \quad (7)$$

where successively $0, 1, \dots 2n-1, 2n$ has to be substituted for k.

In the indicated way we shall get the result

$$\begin{vmatrix} x_0 & x_1 & x_2 \dots & x_{n-1} & x_n \\ x_1 & x_2 & x_3 & x_n & x_{n+1} \\ x_2 & x_3 & x_4 \dots & x_{n+1} & x_{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n-1} & x_n & x_{n+1} \dots & x_{2n-2} & x_{2n-1} \\ x_n & x_{n+1} & x_{n+2} \dots & x_{2n-1} & x_{2n} \end{vmatrix} = 0 \ . \ (8)$$

Likewise in this general case the left hand member of this equation represents an invariant of the binary form of the $2n^{\text{th}}$ degree in $(-\lambda)$, which made equal to nought indicates the osculating space belonging to the point L of λ. In symbols this invariant is indicated by $\underset{n+1}{\Pi}(ab)^2$ where Π is the general sign of multiplication and where the index $n+1$ points to the fact, that the multiplication must be extended to the $\frac{1}{2}n(n+1)$ factors $(ab)^2$ which can be formed of $n+1$ set of coefficients $a, b, c, \dots$ [1]).

By substituting the values (7) in the equation of the osculating space we find

$$\sum_{l=1}^{l=n} p_l (\lambda - \lambda_l)^{2n} = 0 \dots\dots\dots \quad (9)$$

that is to say:

[1]) Probably the general notation $\underset{n+1}{\Pi}(ab)^2$ makes its first appearance here. At least I found everywhere the notation in the form of a determinant and nowhere a symbolic representation nor a reduction to transvectants.

"By taking on the normal curve C_{2n} the n points $L_1, L_2, \dots L_n$ "arbitrarily we determine on it an involution I^{n-1}_{2n} of the dimension "n—1 and the order $2n$, of which each of those n points taken $2n$ "times forms a group. The osculating spaces belonging to the $2n$ "points of any group of that involution intersect in a point A of "the linear space S^{n-1}_1, containing the n given points; if this group "describes the involution I^{n-1}_{2n} the point A generates in S^{n-1}_1 a linear "system in projective correspondence with I^{n-1}_{2n}."

As far as I am aware of up to now a polydimensional interpretation suiting all values of n is known of three general invariants, namely of the discriminant D, of the invariant $(ab)^{2n}$, and of the invariant of SYLVESTER dealt with here. If $x^{n}_{-\lambda}$ is again the equation of the osculating space of the normal curve $x_k = \lambda^k$, $(k = 0, 1, \dots n)$ in the space with n dimensions, corresponding to the parametervalue λ, then $D = 0$ represents as is known the curved space $S^{n-1}_{2(n-1)}$ with n—1 dimensions of the order $2(n—1)$ which is enveloped by the osculating space if λ varies; leaving alone the supposition $n = 2$, which has no sense, we get that n=3 gives in the ordinary space the developable surface having the cubic normal curve of that space as cuspidal line. According to CLIFFORD (l. c.) $(ab)^{2n} = 0$ is in the space S^{2n}_1 the quadratic curved space S^{2n-1}_2 with $2n$—1 dimensions representing the locus of the point lying in a space S^{2n-1}_1 with the points of contact of the $2n$ osculating spaces of the normal curve of that space S^{2n}_1 passing through this point; whilst the corresponding invariant $(ab)^{2n-1}$ of the normal curve of the space S^{2n-1}_1 vanishes identically and the indicated particularity presents itself there, compare the case of the skew cubic in our space, for any point.

For the case $n = 4$ the invariant $(ab)^4 = 0$ is identical with $i = 0$ (CLEBSCH–LINDEMANN l. c.) and at the same time the condition that the four points of contact of the osculatingspaces through any point of the locus form an equianharmonic quadruple. Moreover D is a linear combination of i^3 and j^2, from which finally ensues that any plane cuts the space $D = 0$ according to a curve of the sixth order, having the six points of intersection with the surface of intersection of both spaces $i = 0$, $j = 0$ for cusps; in each of these points the section of the plane with the osculating space of $j = 0$ form the cuspidal tangent. As is known the space $D = 0$, also by the aid of its double surface $i = 0$, $j = 0$, divides the space S^4 into three parts containing the points for which the number of the real osculating spaces passing through them is successively 4, 2 and 0.

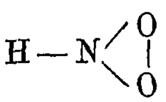
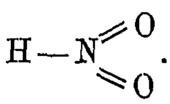
Physics. — Prof. HAGA made, both on behalf of himself and Dr. C. H. WIND a communication: „*On the deflexion of X-rays*".

Deflexion of X-rays was proved on the experiment being arranged as follows:

The RÖNTGEN-tube was placed behind a slit 1 cm. high and 14 microns wide; at 75 cm. from the latter was the diffraction slit, which gradually diminished in width from 14 to about 2 microns. The photographic plate was placed at 75 cm. from the diffraction slit. Time of exposure from 100 to 200 hours. The image of the slit first became narrower and then showed an unmistakable broadening. From the width of the part of the diffraction slit corresponding to this broadening and the character of the broadening an estimination can be made of the wavelength. It appeared that X-rays exist of about 0.1 to $2^1/_2$ ANGSTRÖM units, comprissing 4 octaves.

(A detailed paper will appear in the Proceedings of the next meeting).

Physiology. — Prof. STOKVIS presented for the Library the inaugural dissertation of Dr. G. BELLAAR SPRUYT: „*On the physiological action of methylnitramine in connection with its chemical constitution.*"

At different occasions our member Prof. FRANCHIMONT exposed in our meetings his views about the chemical structure of nitramines, especially of methylnitramine. Till yet the question about the intimate chemical constitution of these compounds, in reference to the manner, in which their nitrogen is linked with the other elements, is an open one. Whereas some authors believe, that the nitrogen of nitramines is linked with hydroxyle, so that the whole compound is a species of nitrite: H—O—N=O, Prof. FRANCHIMONT rejects this view, and considers it linked in a cyclical way, for instance  or . As Prof. FRANCHIMONT considered it of some value to study the physiological action of nitramines, to the aim of throwing more light on the open

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